WILLIAM F. BRADBURY.

AN ELEMENTARY TREATISE

ON

PLANE TRIGONOMETRY.

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AN ELEMENTARY TREATISE

ON

PLANE TRIGONOMETRY

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PREFACE.

IN the present work, those parts of the subject of Trigonometry are treated, which are usually taught in schools. Although the main object of elementary Trigonometry is to provide methods for the solution of triangles, many developments are given which are not absolutely necessary for this purpose; students who wish to push on as rapidly as possible to the solutions of triangles may omit a good deal of the first part of the book. Those Articles and Chapters which are marked with an asterisk may with advantage be omitted by all students, on a first reading. A very large number of examples is given, but it is not by any means desirable that the student should work them all as he goes along, he should rather, on the first and second readings, try only the earlier portions of each set of examples. The Miscellaneous Examples at the end of the work will be found to be of considerably greater difficulty than those in the text.

The authors desire to acknowledge their indebtedness to the Cambridge University Local Examination papers of recent years; the questions in the examination papers at the end of the book are mainly drawn from this source.

CONTENTS.

CHAPTER I.

THE MEASUREMENT OF ANGLES.

1. THE primary object of Plane Trigonometry is the measurement of triangles.

A triangle has three sides and three angles, and supposing the magnitude of any three of these six "parts," as they are called, to be given, it is usually possible to determine the magnitudes of the remaining three parts.

This is called *solving* the triangle.

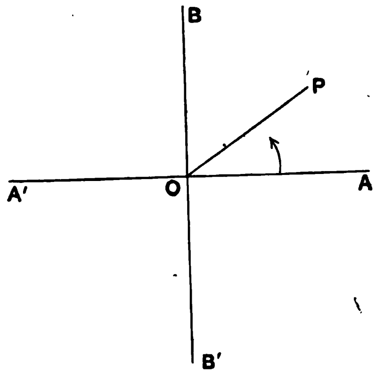
Notice that of the three given parts one at least must be a side.

Plane trigonometry in an extended sense includes the investigation of the properties of certain "trigonometrical ratios," which are introduced in the first instance for the above-mentioned purpose.

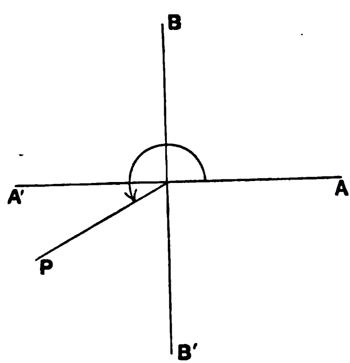
2. How angles are generated in trigonometry.

Let a line OP starting from the position OA turn round O in a direction opposite to that in which the hands of a watch revolve. The *amount of turning* undergone by OP measures *the angle* bounded by OA and OP. The figure represents a particular position of OP. As OP goes on revolving it will reach OB, and has then turned through a right angle.

We may suppose the turning to continue and the revolving line to reach OA'; it has then turned through two right angles.



The angles considered by Euclid are all less than two right angles, but in Trigonometry this limitation as to size of angles is *removed.* For we may suppose OP to reach the position indicated in the second figure; where the angle which has been described by OP is greater than two right angles.



For convenience an arrowhead is sometimes used, as in the figures, to show the angle turned through by OP.

As OP continues its movement it will again reach the position OA from which it started; four right angles have then been described.

This turning may go on indefinitely, and an angle of *any magnitude* may thus be arrived at. Each time a complete turn is made, four right angles have been described.

If we had made OP turn in the opposite direction, viz. in that of the hands of a watch, we should have got another set of angles, all described in the *clockwise* direction.

To distinguish between these two ways of generating angles, it is agreed to call angles described in the *first* of the above, or contra-clockwise direction, *positive*; angles described in the clockwise direction are said to be *negative*.

As an illustration of the generation of angles of any magnitude, think of the angle generated by the large hand of a clock. Each hour this hand turns through four right angles and keeps no record of the number of these complete turns; this, however, is done by the small hand, which only turns through one-twelfth of four right angles in the hour, and thus enables us to measure the angle turned through by the large hand in any time less than twelve hours. For instance, at a quarter past three the large hand has, since noon, turned through thirteen right angles in the negative direction.

Ex. 1. Give figures showing the position of OP after having described

(i) 3 right angles,

(ii) $7\frac{1}{2}$ right angles,

(iii) $1\frac{1}{2}$ right angles negatively.

Ex. 2. Through how many right angles has the large hand of a clock turned since the previous noon at

(i) half past seven p.m.? *Ans.* 30.

(ii) three o'clock a.m.? *Ans.* 60.

3. Coterminal angles.

If in the first figure, OP is the position in which the revolving line stops, it may have arrived at that position by turning through any one of a set of angles infinite in number.

For, to do so, starting from OA it might describe merely the Euclidean angle AOP; or it might from this position then proceed through four right angles and *then* stop, or through eight right angles, sixteen right angles, and so on, before stopping.

Angles bounded by the same two lines OA, OP may be called *coterminal* angles; the angles just mentioned are coterminal.

These angles are all positive; a set of negative angles coterminal with them is obtained, by making the revolving line turn from OA to OP in the *negative* direction, and then stop, or by making it turn *additionally* through four right angles in the negative direction, before stopping, and so on.

These sets of positive and negative angles are clearly got by adding to the Euclidean angle AOP taken with its proper sign, a multiple of four right angles; the multiple being positive for the first set, and negative for the second.

These sets of angles having the same bounding lines will be denoted by the symbol (OA, OP), for the sake of convenience.

4. How angles are measured numerically.

In order to represent angles by numbers we must fix upon some unit angle, then any angle will be represented by the number of times it contains this unit angle.

The unit generally used is the *degree*, which is the one-ninetieth part of a right angle.

Each degree is divided into sixty equal parts called *minutes*, and each *minute* into sixty *seconds*.

Minutes and seconds are introduced in order to avoid using fractions of a degree.

An angle of d degrees is represented by d°, an angle of m minutes by m', and an angle of n seconds by n''; thus the angle $d^\circ\ m'\ n''$ means an angle which contains

$$d \text{ degrees} + m \text{ minutes} + n \text{ seconds}.$$

It is equal to $\dfrac{d}{90} + \dfrac{m}{90 \times 60} + \dfrac{n}{90 \times 60 \times 60}$ of a right angle.

This method of numerical measurement of angles is called the sexagesimal system.

Ex. 1. How many degrees are there in, half a right angle, one-third of a right angle, three-fourths of two right angles respectively? *Ans.* 45°, 30°, 135°.

Ex. 2. Express in the sexagesimal system

One-twelfth of a right angle. *Ans.* 7° 30′.

One-sixteenth of two right angles. *Ans.* 11° 15′.

Ex. 3. How many degrees does the large hand of a watch describe in a minute of time? *Ans.* 6.

5. It has been proposed to use the decimal system of measurement of angles. In this system the right angle is divided into 100 parts, called *grades*, each grade into 100 minutes, each minute into 100 seconds.

An angle containing g grades, m minutes, n seconds is then written $g^g\, m^{\backprime}\, n^{\backprime\backprime}$.

This system has never been generally adopted.

The sexagesimal and decimal systems are sometimes called respectively the English and the French systems.

Observe that angles expressed in the French system can at once be put in the form of decimals of a grade; e.g. $73^g\, 15^{\backprime}\, 7^{\backprime\backprime} = 73 \cdot 1507$ grades.

Express as decimal fractions of a right angle

(i) $15^g\, 19^{\backprime}\, 17^{\backprime\backprime}$. *Ans.* ·151917.

(ii) $10^g\, 7^{\backprime}\, 16^{\backprime\backprime}$. *Ans.* ·100716.

(iii) $29^g\, 13^{\backprime}\, 6^{\backprime\backprime}$. *Ans.* ·291306.

To find the number of grades contained in an angle which is given in degrees, or vice versa, we proceed as follows :—

Let d be the number of degrees in the angle,

g .... grades..................

Then the ratios $\frac{d}{90}$, $\frac{g}{100}$, being each equal to the ratio of the given angle to a right angle, must be equal to each other, thus

$$\frac{d}{90} = \frac{g}{100}, \quad \text{or} \quad d = \frac{9}{10} g, \quad g = \frac{10}{9} d$$

whence if d be given, g can be found, and conversely.

Ex. 1. Find the number of grades in an angle of 60°.

If g is the required number of grades

$$g = \tfrac{10}{9} \,.\, 60 = \tfrac{10}{3} \,.\, 20 = 66\tfrac{2}{3}.$$

Ex. 2. How many degrees are there in 150 grades?

If d is the required number of degrees

$$d = \tfrac{9}{10} \,.\, 150 = 135.$$

Ex. 3. Express in decimal measure the angle

$$16^\circ\ 9'\ 17''.$$

We must first express the minutes and seconds as a decimal of a degree, as follows;

$$\begin{array}{r|l} 60 & 17 \\ \hline 60 & 9{\cdot}28333\ldots\&c. \\ \hline & {\cdot}15472\ldots\&c. \end{array}$$

The angle is therefore 16·15472 degrees nearly.

To find the number of grades multiply this number by $\frac{10}{9}$, thus

$$\begin{array}{r|r} 9 & 16{\cdot}15472 \\ \hline & 1{\cdot}79496 \\ & 10 \\ \hline & 17{\cdot}9496 \end{array}$$

Ans. $17^g\ 94^{\backprime}\ 96^{\backprime\backprime}$.

Ex. 4. Express in sexagesimal measure $47^g\ 5^{\backprime}\ 92^{\backprime\backprime}$.

The angle contains 47·0592 grades, to find the number of degrees, multiply by $\frac{9}{10}$, thus;

$$\begin{array}{r|r} & 47{\cdot}0592 \\ & 9 \\ \hline 10 & 423{\cdot}5328 \\ \hline & 42{\cdot}35328 \end{array}$$

therefore the number of degrees is 42·35328.

We must now convert ·35328 of a degree into minutes and seconds.

$$\begin{array}{r|r} & {\cdot}35328 \\ & 60 \\ \hline 21 & {\cdot}1968 \\ & 60 \\ \hline & 11{\cdot}808 \end{array}$$

Ans. $42^\circ\ 11'\ 11''{\cdot}808$.

EXAMPLES. I.

1. Express in decimal (French) measure

(1)	45°.	(5)	$3^\circ\,18'\,27''$.	(9)	$7'\,12''$.
(2)	18°.	(6)	$53^\circ\,15'\,9''$.	(10)	$5^\circ\,9'\,14''$.
(3)	27°.	(7)	$108^\circ\,20'\,15''$.	(11)	$173^\circ\,16'\,4''$.
(4)	99°.	(8)	$12^\circ\,17'\,11''$.	(12)	$27^\circ\,10'\,16''$.

2. Express in sexagesimal (English) measure.

(1)	25^g.	(5)	$13^g\,18^{`}\,25^{``}$.	(9)	$10^g\,15^{`}\,73^{``}$.
(2)	10^g.	(6)	$125^g\,10^{`}\,19^{``}$.	(10)	$69^g\,15^{`}\,8^{``}$.
(3)	15^g.	(7)	$35^g\,50^{`}$.	(11)	$1^g\,1^{`}\,1^{``}$.
(4)	1^g.	(8)	$10^{``}$.	(12)	$20^{`}\,12^{``}$.

3. The sum of two angles is 20°, their difference is 20^g; find each angle in degrees.

4. Two angles are as $1 : 2$, and the sum of the number of degrees in the one and the number of grades in the other is 58, find the angles.

5. Find the magnitude of the angles of an isosceles triangle which has each of the base angles double of the vertical angle.

6. Find the number of degrees in the vertical angle of an isosceles triangle each of whose base angles is two and a half times the vertical angle.

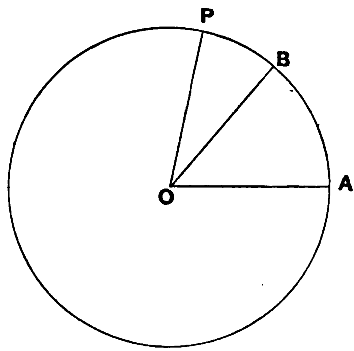
7. If an angle is expressed in French minutes show that it is expressed in English minutes by multiplying by ·54.

8. If an angle is expressed in French seconds show that it is expressed in English seconds by multiplying by ·324.

9. Find the ratio of $24^\circ\,42'\,15''$ to $123{\cdot}1125$ grades.

6. The circular measurement of angles.

For practical purposes the sexagesimal method is almost universally used, but for theoretical work it is more convenient to take a different unit angle.



In any circle of centre O, let AB be an arc whose length is equal to the radius of the circle; we shall show in the next article that the angle AOB is *constant* (or of the same magnitude in all circles).

This angle is taken as the unit of what is called *circular measure*, and is named a *radian*.

The circular measure of any angle AOP will then be the number of times it contains the radian, or the ratio of AOP to AOB.

For the sake of convenience we shall in general use English letters A, B, C—for angles measured in degrees, and Greek letters α, β, θ—for angles measured in radians.

7. In order to prove that the radian is a fixed angle the following theorems are necessary.

(*a*) In the same circle, the lengths of different arcs are to one another in the same ratio as that of the angles which

those arcs subtend at the centre of the circle. This theorem is contained in Euc. VI. 33.

(*b*) The ratio of the radius of a circle to its circumference is the same for all circles. This is proved in Article 8.

From (*a*) it follows that

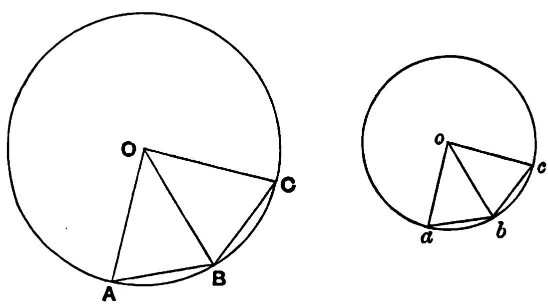
$$\frac{\text{arc } AB}{\text{whole circumference of circle}} = \frac{\angle AOB}{4 \text{ right angles}},$$

but since arc AB = radius of circle, it follows from (*b*) that the first ratio is the same for all circles, or constant;

hence $\angle AOB$ is of constant magnitude.

8. Proof that the ratio of the radius of a circle to its circumference is constant.

Let ABC, abc, be any two circles in each of which a regular polygon of n sides is inscribed. Join the centres O, o to the vertices of the polygons.



Denote the perimeters or the sums of the sides of the polygons by P, p; then since the sides AB, ab are $\frac{1}{n}$th of the respective perimeters,

$$P = n \,.\, AB, \qquad p = n \,.\, ab.$$

Again, the triangles OAB, oab are similar, therefore

$$\frac{OA}{oa} = \frac{AB}{ab} = \frac{n . AB}{n . ab} = \frac{P}{p}.$$

Now let the number of sides be increased indefinitely, P and p approach and ultimately become equal to the circumferences of the circles, therefore

$$\frac{OA}{oa} = \frac{\text{circumference of circle } ABC}{\text{circumference of circle } abc},$$

or $$\frac{\text{radius of } ABC}{\text{circumference of } ABC} = \frac{\text{radius of } abc}{\text{circumference of } abc}.$$

9. The ratio of the circumference of a circle to its diameter, thus proved constant, is denoted by the Greek letter π. So that if r is the radius of a circle

$$\text{circumference of circle} = 2\pi r.$$

The value of π can only be obtained in the form of an infinite non-recurring decimal whose value to eight places is 3·14159265.

For many purposes it is sufficient to use for π the numbers $\frac{22}{7}$ or $\frac{355}{113}$, which agree with the correct value of π as far as two and six places of decimals respectively.

10. Magnitude of the Radian.

We saw in Article 7, that in any circle,

$$\frac{\text{radius}}{\text{circumference}} = \frac{\text{radian}}{\text{4 right angles}},$$

but $\dfrac{\text{circumference}}{\text{diameter}} = \pi$, therefore $\dfrac{\text{radius}}{\text{circumference}} = \dfrac{1}{2\pi}$,

hence $\dfrac{1}{2\pi} = \dfrac{\text{radian}}{\text{4 right angles}}$, or radian $= \dfrac{2}{\pi} \times$ right angle.

Using for π the approximate value 3·1415927, we get as the value of the radian, $57^\circ\ 17'\ 44''{\cdot}81$.

Since a right angle $=\frac{\pi}{2}\times$ radian, the circular measure of a right angle is $\frac{\pi}{2}$, and therefore that of two right angles is π, that of four right angles is 2π.

11. To express in degrees an angle given in circular measure.

Let θ be the number of radians in the given angle, the number of degrees being d; then

$$\frac{\theta}{\pi}=\frac{d}{180};$$

for each ratio expresses what part the given angle is of two right angles.

If the angle be given as $d^\circ\ m'\ n''$, its circular measure is

$$\left(d+\frac{m}{60}+\frac{n}{3600}\right)\frac{\pi}{180}.$$

Comparing with a previous result, we see that

$$\frac{\theta}{\pi}=\frac{d}{180}=\frac{g}{200}.$$

Ex. 1. Express in circular measure the angle

$$10^\circ\ 15'\ 27''.$$

Convert the minutes and seconds into the decimal of a degree, we thus get 10·2575°.

Hence the angle contains $\dfrac{\pi\times 10{\cdot}2575}{180}$ radians.

Ex. 2. Express in sexagesimal measure the angle whose circular measure is $\frac{2}{7}$.

Here θ is $\frac{2}{7}$, and therefore $d=\frac{2}{7}\times\dfrac{180}{\pi}$.

Taking for π the approximate value $\frac{22}{7}$ we get for d the value $\frac{180}{11} = 16\frac{4}{11}$. The angle is thus $16\frac{4}{11}^\circ = 16^\circ\, 22'$ nearly.

EXAMPLES. II.

1. Express in sexagesimal measure the angle whose circular measure is

(1) $\dfrac{\pi}{3}$.	(5) $\dfrac{3\pi}{2}$.	(9) $\cdot 02$.
(2) $\dfrac{\pi}{10}$.	(6) $\dfrac{4\pi}{3}$.	(10) $1\frac{1}{2}$.
(3) $\dfrac{\pi}{12}$.	(7) 1.	(11) $\frac{3}{5}$.
(4) $\dfrac{\pi}{54}$.	(8) 10.	(12) $\dfrac{1}{\pi}$ (take $\pi = \frac{22}{7}$).

See Article 10.

2. Express in circular measure the angles

(1) 45°.	(6) 1°.	(11) $3^\circ\, 17'\, 22{\cdot}2''$.
(2) 60°.	(7) $15'$,	(12) $23^\circ\, 7'\, 30''$.
(3) 150°.	(8) $15^{\backprime}$.	(13) $67^{g}\, 18^{\backprime}\, 86^{\backprime\backprime}$.
(4) 50^{g}.	(9) 2000^{g}.	(14) $93^{g}\, 65^{\backprime}\, 40^{\backprime\backprime}$.
(5) 5°.	(10) $15^\circ\, 26'\, 6''$.	(15) π°.

12. Length of the arc of a circle.

The circular measure of any angle AOP *subtended by the arc* AP *is equal to the ratio* $\dfrac{\text{arc } AP}{\text{radius of circle}}$.

For this ratio is, by Art. 7, $\dfrac{\text{arc } AP}{\text{arc } AB}$, which by Euc. VI. 33 is equal to $\dfrac{\angle AOP}{\angle AOB}$, which is the circular measure of $\angle AOP$.

Hence if θ is the circular measure of $\angle AOP$, and r the radius of the circle

$$\theta = \frac{\text{arc}}{r},$$

hence $$\text{arc} = r\theta,$$

thus, the length of the arc of a circle is $r\theta$, where θ is the number of radians in the angle subtended by the arc at the centre.

If θ is 2π, or the angle four right angles, we get the whole circumference, thus as already given in Article 9,

$$\text{circumference} = 2\pi r.$$

Ex. 1. Find the length of the circumference of the circle whose radius is 6 feet, taking $\frac{22}{7}$ as the value of π.

Since circumference $= 2\pi r$, we have as the required length of circumference, $2 \cdot \frac{22}{7} \cdot 6$ feet $= 37\frac{5}{7}$ feet.

Ex. 2. The arc of a circle is 5 inches, the radius of the circle is 8 yards; find the circular measure of the angle subtended by the arc at the centre.

Since circular measure of an angle $= \frac{\text{arc}}{\text{radius}}$, the circular measure required is $\frac{5}{8 \times 3 \times 12}$. Thus the angle contains $\frac{5}{288}$ radians.

Ex. 3. A fly-wheel makes 300 revolutions in one second; find as a decimal of one second the time it takes to turn through an angle of one radian.

The wheel in one second turns through 300 times four right angles, or $300 \times 2\pi$ radians;

it therefore turns through one radian in $\frac{1}{300 \times 2\pi}$ of a second.

Inserting for π the value $\frac{22}{7}$, we get as the required time

$$\frac{1}{300 \times \frac{44}{7}} = \frac{7}{13200} = \cdot 00053 \text{ of a second, nearly.}$$

Ex. 4. Find to three places of decimals the radius of a circle in which an arc 15 inches long subtends at the centre an angle $71^\circ\ 36'\ 3{\cdot}6''$.

We have seen that radius $= \dfrac{\text{arc}}{\text{circular measure of angle}}$;

hence required radius $= \dfrac{15}{\frac{\pi}{180}\left(71 + \frac{36}{60} + \frac{3\cdot 6}{3600}\right)}$ inches.

Now $\dfrac{36}{60} + \dfrac{3\cdot 6}{3600}$ reduced to a decimal is $\cdot 601$,

$$\text{thus radius} = \frac{15 \times 180 \times 7}{22 \times 71\cdot 601} \text{ inches}$$

$$= 11\cdot 998 \text{ inches, nearly.}$$

EXAMPLES. III.

π is assumed to be $\frac{22}{7}$ unless otherwise stated.

1. Find the length of the circumference of a circle whose radius is 100 yards.

2. Assuming the earth's circumference to be $25{,}142\frac{6}{7}$ miles, find its radius.

3. Find the length of the arc which subtends an angle whose circular measure is 4, at the centre of a circle of radius 12 feet, 3 inches.

4. Find the circular measure of the angle subtended by an arc 6 inches long, at the centre of a circle whose radius is 11 feet.

5. If an arc 15 yards long contains 3 radians, what is the radius of the circle?

6. If the arc of a circle of radius r subtends d degrees at the centre, show that its length is $r \cdot \frac{\pi d}{180}$.

7. Find the radius of the circle in which an arc 6 inches long subtends 45° at the centre.

8. Equal arcs are taken in two circles whose radii are as 1 : 5; the arc in the smaller circle subtends at the centre, an angle of 30°, what angle is subtended by the other arc?

9. If an arc subtends 20° 17′ at the centre of a circle whose radius is 6 inches, find what angle it would subtend in a circle whose radius is 8 inches.

13. Area of a circle and of a sector.

The area of a triangle is known to be measured by half the product of the lengths of the base and altitude, or shortly, $\frac{\text{base} \times \text{altitude}}{2}$.

Consider a regular polygon inscribed in a circle; then using the figure of Article 8, area of polygon $= n$ times area of triangle AOB,

$$= n \times \frac{AB \times \text{altitude of triangle } AOB}{2};$$

but $$n \times AB = P,$$

$\therefore$ area of polygon $= \frac{1}{2}P \times$ altitude of each triangle.

Now if, as in Article 8, the number of sides be indefinitely increased, P becomes the circumference of circle, or $2\pi r$, and the altitude of each triangle becomes the radius, or r;

$\therefore$ *area of circle* $= \frac{1}{2} \cdot 2\pi r \cdot r = \pi r^2$.

Again, $\frac{\text{area of sector of angle } \theta}{\text{area of circle}} = \frac{\theta}{2\pi}$, [Euc. VI. 33.

$\therefore$ *area of sector* $= \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$.

EXAMPLES. IV.

1. A line turning about a point makes three complete revolutions and a sixth, what are the measures of the angle it has turned through, in degrees and in circular measure?

2. Show that the angles turned through per minute by the hour and minute hands of a watch are, respectively, $30'$ and $6°$.

3. What is the length of the arc which subtends $81°$ at the centre of a circle whose radius is 14 feet?

4. Two radii are drawn from the centre of a circle so as to include an angle of $60°$, what is the ratio of the length of arc they intercept, to its chord?

5. Find approximately the area of the circle whose radius is 10 feet.

6. Find the area of a sector of a circle, radius 12 inches, whose angle is $22\frac{1}{2}°$.

7. A semi-circle is divided into two sectors whose angles are as $1:2$; find the areas of the sectors.

8. The area of a circle is 100 sq. yards, what is the circular measure of the angle subtended at the centre, by an arc of 10 feet?

9. Find to one decimal place, the side of a square whose area is equal to that of a circle whose radius is 15 feet.

10. Find the length of a circular railway curve which subtends $22\frac{1}{2}°$ at the centre, the radius being a mile.

11. If the radius of a circle is 25 feet, find to 3 places of decimals the length of arc subtending $3°$ at the centre, taking π to be $3{\cdot}1416$.

12. A water-wheel whose diameter is 12 feet makes 30 revolutions per minute; give approximately the number of miles per hour travelled by a point on the rim.

13. In a parallelogram the number of degrees in one of the angles is to the number of grades in another as $9:8$. Find the angles.

14. The difference of two angles is $10°$, and the circular measure of their sum is 2. Find the circular measure of each angle.

[Assume x and y to be the number of radians in each angle, then

$$\frac{180}{\pi}(x-y)=10, \quad x+y=2.]$$

15. If the number of degrees in an angle is equal to the number of grades in its complement, show that the circular measure of the angle is $\frac{5\pi}{19}$.

16. Express in grades the sum of the angles of a quindecagon.

17. The angles of a triangle are as $1:2:3$, express them in degrees and grades.

[The angles are in arithmetical progression; hence we may assume $x-y$, x, $x+y$ to be the angles expressed in degrees, then since their sum is $=180°$, $3x=180$, or $x=60$; also $\frac{x-y}{x}=\frac{1}{2}$, which determines y.]

18. The three angles of a triangle are in arithmetical progression and the greatest is double the smallest. Find each angle in degrees.

19. The angles of a triangle are in arithmetical progression. The number of grades in the greatest is to the

number of degrees in the sum of the other two as 10 : 11. Find the angles in degrees.

20. In an isosceles triangle the number of degrees in the vertical angle is equal to the number of grades in each base angle. Find the vertical angle.

21. Find the angle in degrees and in grades of

(i) a regular pentagon,
(ii) „ „ hexagon,
(iii) „ „ decagon.

22. If such an angle be taken as the unit that an angle of 60° is represented by 10, find this unit angle.

23. Find the unit when the measure of 120° is $\frac{15}{16}$.

24. Taking for unit angle the angle between two consecutive sides of a regular hexagon, find how many units of the kind there are in

(i) a right angle,
(ii) a radian.

25. Find the ratio of the angle $\frac{\pi}{6}$ to the angle 5°.

[Express the first angle in degrees.]

26. Find the ratio of $10^g\ 12^{`}\ 50^{``}$ to $\frac{\pi}{10}$.

27. What is the ratio of $6^g\ 66^{`}\ 66{\cdot}\dot{6}^{``}$ to 18°?

28. Express in sexagesimal measure the angle whose circular measure is 1·04719, and in circular measure an angle of 125°, taking for π the value 3·14159.

29. If $1620 . g + 1800 . d = 1$, where g and d are the numbers of grades and degrees respectively in an angle, find the magnitude of the angle in English seconds.

30. The diameter of the earth being 8000 miles, what is the distance from the equator measured on the earth's surface, of a place in latitude 30°?

31. If the perimeter of a sector of a circle is equal to the length of the arc of a semi-circle having the same radius, find the angle of the sector in degrees, minutes and seconds.

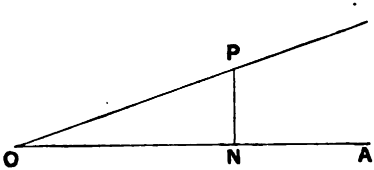
32. In a right-angled triangle the acute angles are $a° \, b'$ and b'; find a and b.

CHAPTER II.

The Trigonometrical Ratios.

14. Having explained the manner in which angular magnitudes are measured, we proceed to define the trigonometrical ratios. In the present chapter we confine ourselves to *acute* angles; in a later chapter we show how the definitions may be extended to angles of any size.

Let P be any point on the line which generates the angle. From P draw PN perpendicular to OA, the other bounding line.



Then denoting AOP by A,

the ratio $\dfrac{NP}{OP}$ is called the *sine* of A, (written $\sin A$),

the ratio $\dfrac{ON}{OP}$ is called the *cosine* of A, (written $\cos A$),

the ratio $\dfrac{NP}{ON}$ is called the *tangent* of A, (written $\tan A$).

We thus have

$$\sin A = \frac{\text{side opposite } A}{\text{hypotenuse}},$$

$$\cos A = \frac{\text{side adjacent to } A}{\text{hypotenuse}},$$

$$\tan A = \frac{\text{side opposite } A}{\text{side adjacent to } A}.$$

There are three other trigonometrical ratios in common use, which are the *inverses* of the sine, cosine, and tangent respectively; they are called the *cosecant*, *secant* and *cotangent*, and are written cosec A, sec A and cot A.

Thus
$$\operatorname{cosec} A = \frac{\text{hypotenuse}}{\text{side opposite } A} = \frac{OP}{NP},$$

$$\sec A = \frac{\text{hypotenuse}}{\text{side adjacent to } A} = \frac{OP}{ON},$$

$$\cot A = \frac{\text{side adjacent to } A}{\text{side opposite } A} = \frac{ON}{NP}.$$

15. The trigonometrical ratios depend only on the angle.

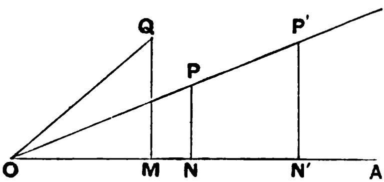
We notice that the trigonometrical ratios do not depend on the position of P on the line OP, for if any other point P' be taken on OP, draw $P'N'$ perpendicular to OA, then the triangles OPN, $OP'N'$ are similar, and therefore

$$\frac{NP}{OP} = \frac{N'P'}{OP'},$$

similarly for the other ratios. Thus the shape of the triangle is known if one of the trigonometrical ratios is known.

If the angle A changes, so also do its trigonometrical ratios.

Let us take a line OQ making with OA an angle AOQ which is greater than AOP, and let OQ be equal to OP. Draw QM perpendicular to OA.



Then $\sin QOM = \dfrac{MQ}{OQ} = \dfrac{MQ}{OP}$, and $\sin POM = \dfrac{NP}{OP}$; but MQ is greater than NP, and therefore $\sin QOM$ is greater than $\sin PON$.

Again $\cos QOM = \dfrac{OM}{OQ} = \dfrac{OM}{OP}$, and $\cos POM = \dfrac{ON}{OP}$; but OM is less than ON, therefore $\cos QOM$ is less than $\cos PON$.

Hence we see that as the angle increases, remaining acute, its sine increases and its cosine diminishes.

Notice that the trigonometrical ratios are *mere numbers* and can therefore be treated as ordinary algebraical quantities; for instance,

$$(\sin A + \cos A)^2 = (\sin A)^2 + 2 \sin A \cos A + (\cos A)^2.$$

Ex. 1. In a triangle ABC having a right angle at C, we are given that $AB = 5$, $BC = 4$, $AC = 3$.

Find the sine and cosine of each acute angle.

Ans. $\frac{4}{5}$, $\frac{3}{5}$; $\frac{3}{5}$, $\frac{4}{5}$.

Ex. 2. The sides of a right-angled triangle are to one another as $25 : 24 : 7$, find all the trigonometrical ratios of the acute angles.

16. Equations connecting the trigonometrical ratios of an angle.

Referring to the definitions just given, it is seen at once that the following relations connect the trigonometrical ratios.

$$\left.\begin{aligned}\cos A \times \sec A &= 1\\ \sin A \times \operatorname{cosec} A &= 1\\ \tan A \times \cot A &= 1\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also

$$\tan A = \frac{\text{side opposite } A}{\text{side adjacent to } A} = \frac{\dfrac{\text{side opposite } A}{\text{hypotenuse}}}{\dfrac{\text{side adjacent to } A}{\text{hypotenuse}}} = \frac{\sin A}{\cos A},$$

thus $\qquad \tan A = \dfrac{\sin A}{\cos A}$

and similarly $\qquad \cot A = \dfrac{\cos A}{\sin A} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$

Equations (2) express the fact that the *tangent* of an angle is the ratio of the sine to the cosine, the *cotangent* is the ratio of the cosine to the sine.

17. Other connecting equations.

Since OPN is a right angle we have by Euclid I. 47,

$$OP^2 = NP^2 + ON^2 \ldots\ldots\ldots\ldots\ldots\ldots (a),$$

$$\therefore\ 1 = \left(\frac{NP}{OP}\right)^2 + \left(\frac{ON}{OP}\right)^2,$$

$$= (\sin A)^2 + (\cos A)^2,$$

or, as it is usually written,

$$1 = \sin^2 A + \cos^2 A.$$

From (a) there follows also that

$$\left(\frac{OP}{NP}\right)^2 = 1 + \left(\frac{ON}{NP}\right)^2,$$

or $$\operatorname{cosec}^2 A = 1 + \cot^2 A,$$

and that $$\left(\frac{OP}{ON}\right)^2 = 1 + \left(\frac{NP}{ON}\right)^2,$$

or $$\sec^2 A = 1 + \tan^2 A.$$

The following three equations are therefore merely different forms of the same theorem:

$$\left.\begin{aligned} \sin^2 A + \cos^2 A &= 1 \\ 1 + \tan^2 A &= \sec^2 A \\ 1 + \cot^2 A &= \operatorname{cosec}^2 A \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(3).$$

EXAMPLES. V.

If a question involves a trigonometrical ratio of A other than $\sin A$ or $\cos A$, $\tan A$ for instance, it is usually best to replace $\tan A$ by $\dfrac{\sin A}{\cos A}$, similarly for $\cot A$. Replace $\sec A$ and $\operatorname{cosec} A$ by $\dfrac{1}{\cos A}$ and $\dfrac{1}{\sin A}$.

From the definitions in Article 14, and equations (1), (2) and (3), prove the following:

1. $\dfrac{1}{\operatorname{cosec}^2 A} + \dfrac{1}{\sec^2 A} = 1.$

2. $\dfrac{1}{\cos^2 A} = 1 + \dfrac{1}{\cot^2 A}.$

3. $\dfrac{1}{\sin^2 A} = 1 + \dfrac{1}{\tan^2 A}.$

4. $\cos^2 A - \sin^2 A = 2\cos^2 A - 1.$

5. $4\cos^2 A - 3 = 1 - 4\sin^2 A.$

6. $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A.$

7. $(\sin A + \cos A)^2 = 1 + 2\sin A \cos A.$

8. $\dfrac{\sin^2 A}{1+\cos A} = 1 - \cos A.$

9. $\tan^2 A \,.\, \cos^2 A + \cot^2 A \,.\, \sin^2 A = 1.$

10. $\left(\dfrac{\tan A}{\sin A}\right)^2 = 1 + \dfrac{\sin^2 A}{\cos^2 A}.$

11. $\dfrac{\tan A}{1 - \tan^2 A} = \dfrac{\sin A \cos A}{\cos^2 A - \sin^2 A}.$

12. $\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \,.\, \operatorname{cosec}^2 A.$

13. $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} = \cos^2 A - \sin^2 A.$

14. $\tan^2 A + \tan^4 A = \dfrac{\sin^2 A}{\cos^4 A}.$

15. $1 - \sin^4 A = \cos^2 A\,(1 + \sin^2 A).$

16. $\tan A + \cot A = \sec A \,.\, \operatorname{cosec} A.$

17. $\sin^3 A \cos A + \cos^3 A \sin A = \sin A \cos A.$

18. $\sin^2 A \cos^2 A + \cos^4 A = 1 - \sin^2 A.$

18. When one trigonometrical ratio is given, to find the other ratios.

The equations just obtained enable us to find any five of the trigonometrical ratios when the sixth is given.

Take as an example the following case:—

having given $\tan A = \sqrt{3},$

we have $\cot A = \dfrac{1}{\tan A} = \dfrac{1}{\sqrt{3}},$

hence $\sec^2 A = 1 + \tan^2 A = 1 + 3 = 4,$

$$\therefore \ \sec A = 2, \quad \text{and} \quad \cos A = \frac{1}{\sec A} = \tfrac{1}{2},$$

also
$$\sin^2 A = 1 - \cos^2 A,$$
$$= 1 - \tfrac{1}{4} = \tfrac{3}{4},$$

$$\therefore\ \sin A = \frac{\sqrt{3}}{2}, \text{ and } \operatorname{cosec} A = \frac{1}{\sin A} = \frac{2}{\sqrt{3}}.$$

At the end of the chapter a *table* is given, showing generally how to express any five of the ratios in terms of the sixth.

EXAMPLES. VI.

In the following cases the value of one trigonometrical ratio is given, it is required to find the values of the other ratios.

1. $\sin A = \frac{1}{2}$.
2. $\cos A = \frac{1}{\sqrt{2}}$.
3. $\tan A = \frac{4}{3}$.
4. $\operatorname{cosec} A = 4$.
5. $\sec A = \frac{61}{11}$.
6. $\cot A = 2 + \sqrt{3}$.

19. The ratio of an angle is equal to the co-ratio of its complement.

If we agree to call either of the trigonometrical ratios $\sin A$ and $\cos A$ the *co-ratio* of the other, so that

$$\begin{cases} \sin A \text{ is the co-ratio of } \cos A \\ \cos A \text{ is the co-ratio of } \sin A, \end{cases}$$

similarly
$$\begin{cases} \tan A \text{ is the co-ratio of } \cot A \\ \cot A \text{ is the co-ratio of } \tan A, \end{cases}$$

and
$$\begin{cases} \sec A \text{ is the co-ratio of } \operatorname{cosec} A \\ \operatorname{cosec} A \text{ is the co-ratio of } \sec A, \end{cases}$$

then any trigonometrical ratio of A is the co-ratio of $90° - A$.

For by the definition of the sine as $\frac{\text{side opposite}}{\text{hypotenuse}}$, we see that the sine of OPN is $\frac{ON}{OP}$, which is equal to $\cos A$, and OPN is $90° - A$,

whence arises the result (which will be proved later for all values of A),

sine of an angle = cosine of its complement, or that

$$\sin(90° - A) = \cos A.$$

In the same way

$$\cos OPN = \frac{NP}{OP} = \sin A,$$

or

$$\cos(90° - A) = \sin A.$$

By similar reasoning,

$$\tan(90° - A) = \frac{ON}{NP} = \cot A$$

$$\cot(90° - A) = \frac{NP}{ON} = \tan A.$$

20. Greatest and least values of the trigonometrical ratios.

The hypotenuse is the greatest side of a right-angled triangle, hence the sine and cosine of an angle are always less than unity, therefore the secant and cosecant are always greater than unity, but the tangent and cotangent may have any values.

21. The table which follows gives us the expression of any five of the trigonometrical ratios in terms of the sixth.

It is obtained by the use of equations (1), (2), and (3), of this Chapter, and should be verified by means of them. To explain the use of the table, take, for instance, the third column; this states, that x being written for $\tan A$, then $\sin A$ is equal to $\frac{x}{\sqrt{1+x^2}}$, $\cos A$ equals $\frac{1}{\sqrt{1+x^2}}$, and so on.

	$\sin A=x$	$\cos A=x$	$\tan A=x$	$\cot A=x$	$\sec A=x$	$\operatorname{cosec} A=x$
$\sin A=$	x	$\sqrt{1-x^2}$	$\dfrac{x}{\sqrt{1+x^2}}$	$\dfrac{1}{\sqrt{1+x^2}}$	$\dfrac{\sqrt{x^2-1}}{x}$	$\dfrac{1}{x}$
$\cos A=$	$\sqrt{1-x^2}$	x	$\dfrac{1}{\sqrt{1+x^2}}$	$\dfrac{x}{\sqrt{1+x^2}}$	$\dfrac{1}{x}$	$\dfrac{\sqrt{x^2-1}}{x}$
$\tan A=$	$\dfrac{x}{\sqrt{1-x^2}}$	$\dfrac{\sqrt{1-x^2}}{x}$	x	$\dfrac{1}{x}$	$\sqrt{x^2-1}$	$\dfrac{1}{\sqrt{x^2-1}}$
$\cot A=$	$\dfrac{\sqrt{1-x^2}}{x}$	$\dfrac{x}{\sqrt{1-x^2}}$	$\dfrac{1}{x}$	x	$\dfrac{1}{\sqrt{x^2-1}}$	$\sqrt{x^2-1}$
$\sec A=$	$\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{1}{x}$	$\sqrt{1+x^2}$	$\dfrac{\sqrt{1+x^2}}{x}$	x	$\dfrac{x}{\sqrt{x^2-1}}$
$\operatorname{cosec} A=$	$\dfrac{1}{x}$	$\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{\sqrt{1+x^2}}{x}$	$\sqrt{1+x^2}$	$\dfrac{x}{\sqrt{x^2-1}}$	x

EXAMPLES. VII.

Prove the following identical equations.

1. $\tan^2 A - \cot^2 A = \sec^2 A - \operatorname{cosec}^2 A.$

2. $\sin\theta = \dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}.$

3. $(1+\tan A)^2 + (1-\tan A)^2 = 2\sec^2 A.$

4. $(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2.$

5. $(1+\cos A)^2 + (1+\sin A)^2 = 3 + 2(\sin A + \cos A).$

6. $\tan^2 A + \cot^2 A = \dfrac{1 - 2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}.$

7. $\dfrac{(\operatorname{cosec}\theta+\sec\theta)^2}{\sec^2\theta+\operatorname{cosec}^2\theta}=1+2\sin\theta\cos\theta.$

8. $\sin\theta(1+\tan\theta)+\cos\theta(1+\cot\theta)=\sec\theta+\operatorname{cosec}\theta.$

9. $(1-\tan^4 A)\cos^2 A+\tan^2 A=1.$

10. $\sin^3 A+\cos^3 A=(1-\sin A\cos A)(\sin A+\cos A).$

11. $(\tan A+\sec A+1)(\tan A-\sec A+1)=2\tan A.$

12. $(\sin\theta+\cos\theta)(\tan\theta+\cot\theta)=\sec\theta+\operatorname{cosec}\theta.$

13. $(1+\tan A+\tan^2 A)(1-\cot A+\cot^2 A)$
$$=\tan^2 A+\cot^2 A+1.$$

14. $\dfrac{\sin^2 A+\cos^2 A-\sec^2 A}{\sin^2 A+\cos^2 A-\operatorname{cosec}^2 A}=\tan^4 A.$

15. $(\tan A+\sec A)^2=\dfrac{1+\sin A}{1-\sin A}.$

16. $\tan A(1-\cot^2 A)+\cot A(1-\tan^2 A)=0.$

17. $\sin^2 A\tan^2 A+\cos^2 A\cot^2 A=\tan^2 A+\cot^2 A-1.$

18. If $\tan A+\sin A=m$, $\tan A-\sin A=n$, show that
$$m^2-n^2=4\sqrt{mn}.$$

19. $\dfrac{1-\sec A+\tan A}{1+\sec A-\tan A}=\dfrac{\sec A+\tan A-1}{\sec A+\tan A+1}.$

20. $\dfrac{1+\operatorname{cosec}A+\cot A}{1+\operatorname{cosec}A-\cot A}=\dfrac{\operatorname{cosec}A+\cot A-1}{\cot A-\operatorname{cosec}A+1}.$

21. $(\sec\theta+\operatorname{cosec}\theta)^2=(1+\tan\theta)^2+(1+\cot\theta)^2.$

22. $2\operatorname{cosec}^2 A=1+\operatorname{cosec}^4 A-\cot^4 A.$

23. $(\sin A+\sec A)^2+(\cos A+\operatorname{cosec}A)^2$
$$=(1+\sec A\operatorname{cosec}A)^2.$$

24. $\tan\theta=\dfrac{\sin\theta+2\sin\theta\cos\theta}{1+\cos\theta+\cos^2\theta-\sin^2\theta}.$

25. $1-\tan^2\theta+\tan^4\theta=\cos^2\theta(1+\tan^6\theta).$

26. $(2-\cos^2 A)(1+2\cot^2 A)=(2+\cot^2 A)(2-\sin^2 A).$

27. $\operatorname{cosec}^4\theta\,(1-\cos^4\theta)-2\cot^2\theta=1.$

28. $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}+\dfrac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}=2\operatorname{cosec}\theta.$

29. Show that $\sin A+\cos A$ is greater than unity when A is less than $90°$.

30. If $\tan\theta=\dfrac{a}{b}$, show that

$$\sin\theta=\frac{a}{\sqrt{a^2+b^2}},\quad \cos\theta=\frac{b}{\sqrt{a^2+b^2}}.$$

31. If $\sin\theta=\dfrac{m^2-n^2}{m^2+n^2}$, find the other ratios.

32. If $\sin\theta=\dfrac{m^2+2mn}{m^2+2mn+2n^2}$, prove that

$$\tan\theta=\frac{m^2+2mn}{2mn+2n^2}.$$

33. Prove that

$$(1+\cos A-\sin^2 A)^2(1-\cos A)^2 + (1+\sin A-\cos^2 A)^2(1-\sin A)^2=\sin^2 A\cos^2 A.$$

34. $\tan\theta+\cot\theta=2\sin\theta\cos\theta+\sin^3\theta\sec\theta+\cos^3\theta\operatorname{cosec}\theta.$

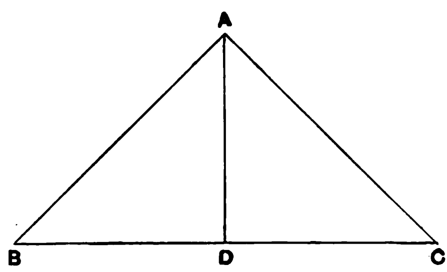
CHAPTER III.

THE TRIGONOMETRICAL RATIOS OF CERTAIN ANGLES.

22. The values of the trigonometrical ratios of a few important angles can be obtained geometrically, as follows:

(i) *For an angle of* 45°, *or* $\frac{\pi}{4}$.

Take an isosceles triangle right-angled at A. Draw AD perpendicular to BC, and therefore bisecting the angle A.



The angle at B is 45°, and since

$$AB = \sqrt{BD^2 + AD^2} = \sqrt{2}BD = \sqrt{2}AD$$

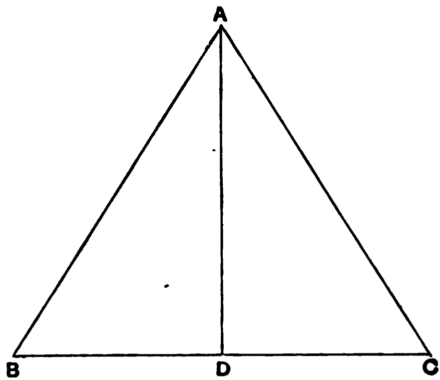
hence $$\sin 45^\circ = \frac{AD}{AB} = \frac{1}{\sqrt{2}},$$

and $$\cos 45^\circ = \frac{BD}{AB} = \frac{1}{\sqrt{2}},$$

$$\therefore \tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = 1.$$

(ii) *For an angle of* 60°, *or* $\frac{\pi}{3}$.

Take an equilateral triangle ABC, each of whose angles is therefore 60°. Draw AD perpendicular to BC.



Then $$\cos 60^\circ = \frac{BD}{AB} = \frac{1}{2},$$

also $$\sin 60^\circ = \sqrt{1 - \cos^2 60}, \qquad \text{[Article 16.}$$

$$= \sqrt{1 - \tfrac{1}{4}} = \frac{\sqrt{3}}{2},$$

and $$\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}.$$

(iii) *For an angle of* 30° *or* $\frac{\pi}{6}$.

In the last figure, the angle BAD is 30°, therefore

$$\sin 30^\circ = \frac{BD}{AB} = \frac{1}{2},$$

also $$\cos 30^\circ = \sqrt{1 - \sin^2 30^\circ} = \sqrt{1 - \tfrac{1}{4}} = \frac{\sqrt{3}}{2},$$

and $$\tan 30^\circ = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$$

The following table contains the above values collected for reference.

	sine	cosine	tangent	cotangent
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1

We notice that 30° and 60° being complementary angles, a ratio of the one is the co-ratio of the other.

EXAMPLES. VIII.

Prove the following:

1. $\tan 30^\circ . \tan 60^\circ = \tan 45^\circ$.
2. $\sin 30^\circ . \cos 60^\circ + \cos 30^\circ . \sin 60^\circ = 1$.
3. $\cos 30^\circ . \cos 60^\circ + \sin 30^\circ . \sin 60^\circ = \frac{\sqrt{3}}{2}$.

4. $\tan 60^\circ . \sin^2 45^\circ = \cos 30^\circ$.

5. $\dfrac{1 - \tan^2 30^\circ}{1 + \tan^2 30^\circ} = \cos 60^\circ$.

6. $\dfrac{1 - \sin 30^\circ}{1 + \tan 45^\circ} = 1 - \cos^2 30^\circ$.

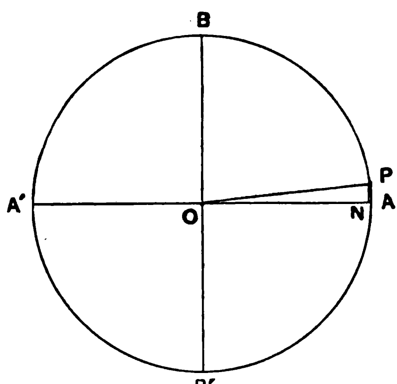
7. $\dfrac{1 + \tan 30^\circ}{1 - \tan 30^\circ} = \dfrac{1 + \sin 60^\circ}{\sin 30^\circ}$.

8. $(1 + \sin 45^\circ + \sin 30^\circ)(1 - \cos 45^\circ + \cos 60^\circ) = \frac{7}{4}$.

9. Show that the acute angle whose sine is $\dfrac{1}{\sqrt{3}}$ lies between 30° and 45°.

10. Show that the acute angle whose cosine is $\dfrac{1}{\sqrt{3}}$ lies between 45° and 60°.

23. The trigonometrical ratios of 0°, and of 90°, or $\frac{\pi}{2}$.



Let us in Article 14, suppose first that OP lies near OA, and next that it lies near OB. Also, for simplicity, suppose

that OP is of fixed length, P then lies on a circle whose centre is O, meeting OA and OB in A and B.

First, when OP lies near OA.

Here NP is small and ON nearly equal to OA.

Therefore

$\sin AOP$, or $\frac{NP}{OP}$, is small; hence $\operatorname{cosec} AOP$ is large,

$\cos AOP$, or $\frac{ON}{OP}$, is nearly unity; hence $\sec AOP$ is nearly unity,

$\tan AOP$, or $\frac{NP}{ON}$, is small; hence $\cot AOP$ is large.

Now suppose OP to approach OA, the $\angle AOP$ approaches $0°$; $\sin AOP$ diminishes continually and ultimately becomes zero, or $\sin 0° = 0$,

hence $\operatorname{cosec} AOP$ increases and ultimately becomes infinitely great, or $\operatorname{cosec} 0° = \infty$.

Cos AOP approaches unity, or $\cos 0° = 1$,

hence $\sec AOP$ approaches unity, or $\sec 0° = 1$.

Tan AOP diminishes and finally becomes zero, or $\tan 0° = 0$,

hence $\cot AOP$ increases and finally becomes infinitely great, or $\cot 0° = \infty$.

Next, when OP lies near OB.

Here ON is small, and NP nearly equal to OB.

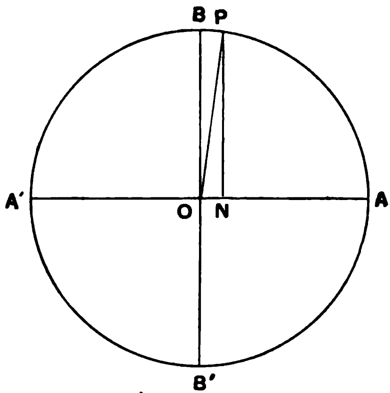
Let OP approach OB; then

$\sin AOP$, or $\frac{NP}{OP}$, approaches unity, or $\sin 90° = 1$,

$\cos AOP$, or $\frac{ON}{OP}$, approaches zero, or $\sin 90° = 0$,

$\tan AOP$, or $\frac{NP}{ON}$, becomes infinitely great, or $\tan 90° = \infty$,

and thus we see that $\begin{cases} \operatorname{cosec} 90° = 1 \\ \sec 90° = \infty \\ \cot 90° = 0. \end{cases}$



The above results are included in the following table.

	sine	cosine	tangent	cotangent	secant	cosecant
0°	0	1	0	∞	1	∞
90°	1	0	∞	0	∞	1

The statements contained in this table and the preceding one should be carefully remembered.

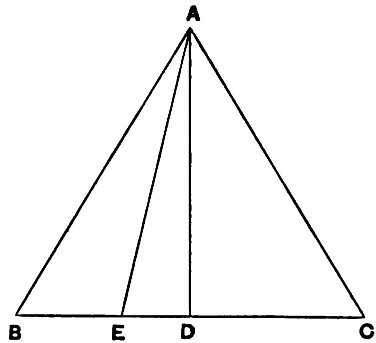
24. The trigonometrical ratios of 15°.

Let ABC be an equilateral triangle, and let AD bisect the angle A; also let AE bisect the angle BAD, then $\angle BAE$ is 15°, and by Euclid VI. 3

$$\frac{DE}{EB} = \frac{DA}{AB} = \sin 60° = \frac{\sqrt{3}}{2}.$$

Therefore

$$1+\frac{EB}{DE}=1+\frac{2}{\sqrt{3}}, \quad \text{or,} \quad \frac{BD}{DE}=\frac{\sqrt{3}+2}{\sqrt{3}},$$



and since $$\frac{DA}{DB}=\tan 60^\circ=\sqrt{3}$$

$$\frac{BD}{DE}\cdot\frac{DA}{DB}=\frac{\sqrt{3}+2}{\sqrt{3}}\cdot\sqrt{3}.$$

$$\therefore\ \cot 15^\circ=\frac{DA}{DE}=2+\sqrt{3}.$$

From this result the other trigonometrical ratios of 15° may be found, see Art. 17.

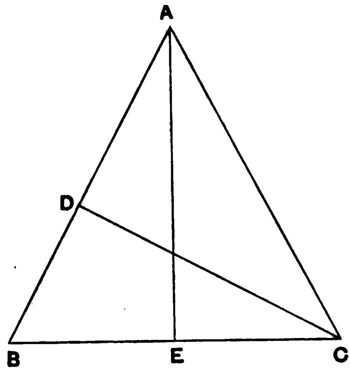
25. The trigonometrical ratios of 36° and 72°.

Take a triangle ABC having each of the angles at the base double of the vertical angle, the base angles are then each 72°, the vertical angle is 36°.

If AB is divided at D so that $AB\,.\,BD=AD^2$, by Euclid IV. 10, we know that $AD=DC=CB$.

Draw AE perpendicular to BC, and denote the ratio of

AD to AB by x; then since $AB(AB - AD) = AD^2$, it follows that $1 - x = x^2$.



Solving the quadratic, we find $x = \dfrac{\pm\sqrt{5} - 1}{2}$; we must take the positive root, hence

$$\frac{AD}{AB} = \frac{\sqrt{5} - 1}{2}, \text{ or } \frac{2BE}{AB} = \frac{\sqrt{5} - 1}{2},$$

thus
$$\cos 72^\circ = \frac{BE}{AB} = \frac{\sqrt{5} - 1}{4}.$$

From this we find

$$\sin 72^\circ = \tfrac{1}{4}\sqrt{10 + 2\sqrt{5}}.$$

Since the triangle ACD is isosceles,

$$\cos 36^\circ = \frac{\frac{1}{2}AC}{AD} = \frac{1}{4}\frac{AC}{CE},$$

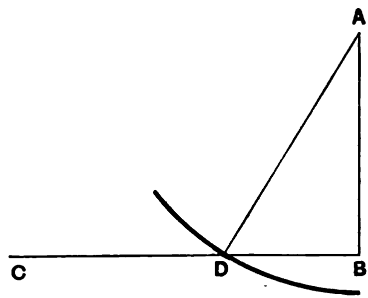
hence
$$\cos 36^\circ = \tfrac{1}{4}\sec 72^\circ = \frac{1}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{4}.$$

The angles 54° and 36° are complementary, and therefore the values of $\sin 54^\circ$ and $\cos 54^\circ$ are found.

26. To find geometrically the angle having a given sine.

Let the given value of the sine be a number n, which is necessarily less than unity.

On any scale measure off a distance AB containing n units of length. Draw BC perpendicular to AB, and with centre A and radius unity describe a circle cutting BC at D.



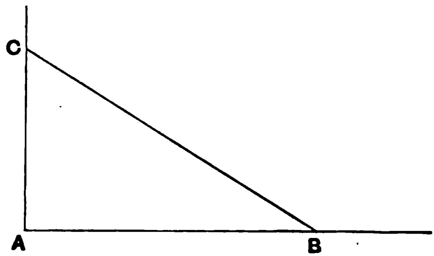
Then, $$\sin ADB = \frac{AB}{AD} = \frac{n}{1}.$$

Hence, ADB is the required angle.

We may use a similar construction to find the angle whose cosine is given.

27. To find geometrically the angle having a given tangent.

Suppose $\frac{a}{b}$ to be the given value of the tangent. Measure on any line a distance AB containing b units of length, and on a perpendicular line measure off a distance AC containing a units of length; join BC.



Then, $$\tan ABC = \frac{AC}{AB} = \frac{a}{b}.$$

Thus ABC is the required angle.

EXAMPLES. IX.

If $A=0°$, $B=30°$, $C=45°$, $D=60°$, $E=90°$, find the values of the following expressions.

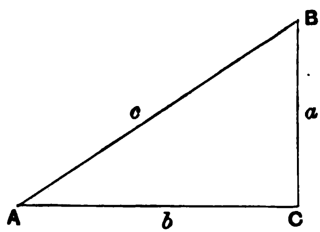
1. $\sin B + \cos C - 1$.
2. $\tan^2 A + \tan^2 B + \tan^2 C$.
3. $\cos A \cos B \cos C + \sin C \sin D \sin E$.
4. $\cot^2 B \operatorname{cosec}^2 C - \tan^2 D$.
5. $\sec A + 2 \sin B + 2 \cos^2 C + \frac{1}{3} \tan^2 D + \operatorname{cosec} E$.
6. $\sin B \cos D + \cos B \sin D - \sin E$.
7. $\cos^2 B + \cos^2 D + \cos^2 E + 2 \cos B \cos D \cos E$.
8. $\sec B (1 + \tan C) - \sin^3 E (\cos C + \sin D \cos B)$.
9. $\dfrac{1 + \tan^2 D}{2 - \tan^2 C} + 3 (\cos A \sin^2 C - \sin D)$.
10. $(\sin B + \sin E)(\cos A + \cos D) - 4 \sin A (\cos C + \sin E)$.
11. $(\tan B + \operatorname{cosec} E + \cot C)(\cos A + \cos E + 1)$.

CHAPTER IV.

THE SOLUTION OF RIGHT-ANGLED TRIANGLES.

28. IF ABC be a triangle, we shall denote the angles BAC, ABC, ACB by A, B, and C, respectively. The sides opposite to the angles A, B and C are denoted by the small letters a, b, and c, respectively.

In the present chapter, we consider the case in which C is a right angle, and therefore c the hypotenuse.



Of the different sides and angles, or *parts*, of a right-angled triangle, one is already known, namely, the right angle; if then two more parts, one at least of which is a side, are known, the others may be calculated.

These two parts may be given in four ways.

(I) Suppose a and b are given;

then A is found from the formula $\tan A = \frac{a}{b}$; also $B = 90° - A$, B is therefore known.

Also $\frac{a}{c} = \sin A$, $\therefore c = a \operatorname{cosec} A$, which finds c after A has been found.

Ex. Given $a = 2$, $b = 2\sqrt{3}$,

$$\tan A = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}, \quad \therefore A = 30°, \text{ hence } B = 60°.$$

Also $$C = 2 \operatorname{cosec} 30° = 4.$$

(II) Suppose c and a are given;

then $\frac{a}{c} = \sin A$, which gives A, and $B = 90° - A$.

Also $\frac{b}{c} = \cos A$, $\therefore b = c \cos A$, which determines b.

Or b might be found from the formula $b^2 = c^2 - a^2$.

Ex. Given $c = 2$, $a = 1$;

$$\sin A = \tfrac{1}{2}, \text{ or } A = 30°, \text{ whence } B = 60°.$$

Also $$b = \sqrt{4 - 1} = \sqrt{3}.$$

(III) Suppose c and A are given;

then B, being equal to $90° - A$, is found.

and $\left\{\begin{array}{l} a = c \sin A \\ b = c \cos A \end{array}\right.$ giving a and b.

(IV) Suppose a and A are given;

then we use $\left\{\begin{array}{l} B = 90° - A \\ c = a \operatorname{cosec} A \\ b = c \cos A. \end{array}\right.$

EXAMPLES. X.

Find the other parts in the following cases; given that

1. $a=3\sqrt{7},\ b=\sqrt{21}.$
2. $a=2,\ b=4.$
3. $c=12,\ a=6.$
4. $c=4,\ a=2\sqrt{3}.$
5. $c=2,\ a=\sqrt{2}.$
6. $c=10,\ A=30°.$
7. $c=12,\ A=15°,$

having given that $\sin 15° = \frac{1}{4}(\sqrt{6}-\sqrt{2})$
$\cos 15° = \frac{1}{4}(\sqrt{6}+\sqrt{2}).$

8. $a=5\sqrt{3},\ A=60°.$
9. $a=10,\ A=45°.$
10. $a=3,\ c=12,$

having given that

$$\sin 14°\ 28'\ 21'' = \cdot 25,$$
$$\cos 14°\ 28'\ 21'' = \cdot 9683.$$

29. Heights and Distances.

We can now show how Trigonometry is applied to find the heights and distances of objects.

In making observations of objects certain instruments are used;

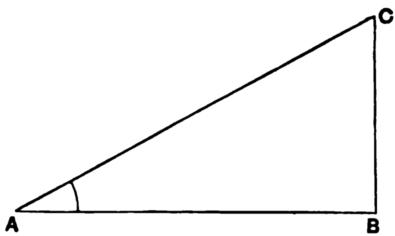
(i) to find the inclination of the line joining the observer's eye to the object, to the horizontal; this angle is called the *elevation* of the object if the object is *above* the observer, and it is called the *depression* if it is *below* the observer.

(ii) To find the angle between the lines joining two distant points to the observer's eye.

The *Sextant* and the *Theodolite* are used for these purposes.

30. The method of calculating will best be understood from the following examples.

Ex. 1. An observer at A, 300 feet distant from the base of a high tower BC, notices that the elevation of B is 30°, find the height of the tower.



Here $AB = 300$, $BAC = 30°$, it is required to find BC.

$$BC = AB \tan BAC,$$

or
$$BC = 300 \times \frac{1}{\sqrt{3}} = 173 \text{ nearly};$$

height of tower = 173 feet, nearly.

N.B. The observer's eye, at A, is supposed to be on a level with the ground.

Ex. 2. A pole is situated on the top of a tower 50 feet high. The elevation of the top of the tower to an observer at A is 30°, the elevation of the top of the pole is 45°. Find the length of the pole, and the distance of the observer from the foot of the tower.

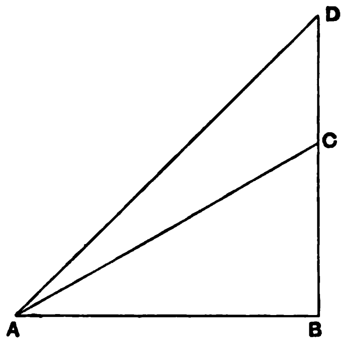
Let the length of the pole CD be x feet, and the distance AB of the observer from the foot of the tower, y feet.

Then we have,

$$\frac{x + 50}{y} = \tan 45° = 1,$$

$$\frac{50}{y} = \tan 30^\circ = \frac{1}{\sqrt{3}},$$

dividing, $$\frac{x+50}{50} = \sqrt{3},$$



which gives

$$x = 50(\sqrt{3}-1) = 36{\cdot}5, \text{ nearly.}$$

Also, $$y = 50\sqrt{3} = 86{\cdot}5, \text{ nearly.}$$

Ans. Height of tower $= 36\frac{1}{2}$ feet, distance of observer from tower $= 86\frac{1}{2}$ feet.

Ex. 3. Instead of supposing the observer's eye on a level with the ground, the observation might have been taken in a standing position.

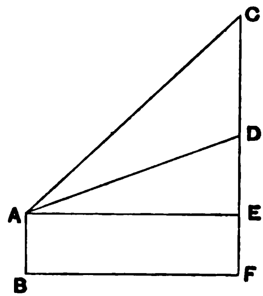
A man 6 feet high observes that the top of a tree has an elevation of 45°, and the point where the branches begin an elevation of 30°. The distance of this last point from the ground is known to be 14 feet, find the height of the tree.

AB representing the observer, draw AE parallel to BF. We are given that $AB = 6$, $DF = 14$, and hence $DE = 8$.

Let x and y be the number of feet in CF and BF respectively.

Then $$\frac{8}{y} = \tan DAE = \tan 30° = \frac{1}{\sqrt{3}}.$$

$$\frac{x-6}{y} = \tan CAE = \tan 45° = 1,$$

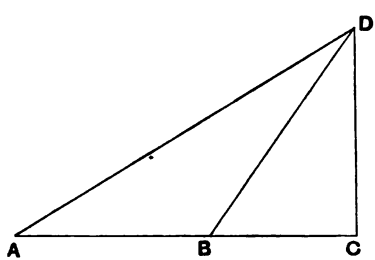


dividing

$$\frac{8}{x-6} = \frac{1}{\sqrt{3}}, \text{ or } x = 6 + 8\sqrt{3} = 19·84, \text{ nearly.}$$

The tree is 19·84 feet high.

Ex. 4. A man observes the elevation of a tower to be 30°, advancing towards the tower a distance of 300 feet he notices the new elevation to be 60°, find the height of the tower.



A is the first position of observer and *B* the second; required the length of *CD*. Let *CD* contain x feet, and *BC* contain y feet, then

$$\frac{x}{y+300} = \tan 30° = \frac{1}{\sqrt{3}}.$$

$$\frac{x}{y} = \tan 60° = \sqrt{3};$$

subtracting, $$\frac{y+300}{x} - \frac{y}{x} = \sqrt{3} - \frac{1}{\sqrt{3}},$$

which gives

$$\frac{300}{x} = \sqrt{3} - \frac{1}{\sqrt{3}}, \text{ or } x = \frac{300 \times \sqrt{3}}{2} = 259\tfrac{1}{2} \text{ nearly.}$$

Ans. Height of tower $= 259\frac{1}{2}$ feet.

The figure might also be used to solve such a question as the following.

A man at *D* on a hill, whose height *CD* is known, sees that the depressions of two objects *A* and *B* in a horizontal straight line lying in a vertical plane through *D*, are 30° and 60° respectively. Find the distance apart of *A* and *B*.

Now $$\frac{AC}{CD} - \frac{BC}{CD} = \cot 30° - \cot 60°.$$

$$= \sqrt{3} - \frac{1}{\sqrt{3}}.$$

From which *AB* is found.

Ex. 5. A man walking along a straight road observes at a milestone a house in a direction of 30° with the road. At the next he sees it in a direction of 60°. Find the distance of the house from the road.

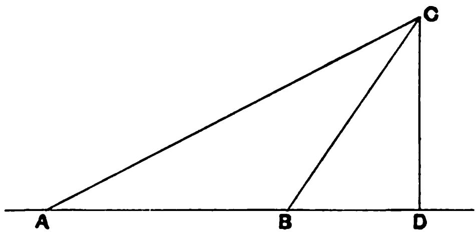
A and *B* being the milestones, it is required to find the distance *CD* of the house from the road.

$$\frac{AD}{CD} = \cot 30° = \sqrt{3},$$

$$\frac{BD}{CD} = \cot 60^\circ = \frac{1}{\sqrt{3}};$$

subtracting

$$\frac{AD - BD}{CD} = \sqrt{3} - \frac{1}{\sqrt{3}}, \text{ or } AB = \frac{2}{\sqrt{3}} CD,$$



hence $$CD = \frac{\sqrt{3}}{2} AB,$$

or the house is distant $\frac{\sqrt{3}}{2}$ of a mile from the road, or ·86 of a mile, nearly.

EXAMPLES. XI.

1. A man who is 200 feet from the base of a chimney sees it at an elevation of 30°. What is its height?

2. A circular pond has a pole standing vertically out of its centre, whose top is 100 feet above the surface. At a point in the circumference the angle subtended by the pole is 60°. Find the radius of the pond.

3. If a stick 2 feet long has a shadow $2\sqrt{3}$ feet in length, find the elevation of the sun.

4. There are two places, A and B, on the banks of a river, exactly opposite each other. A line AC, of

length 50 yards, is measured at right angles to AB, and at C the angle subtended by AB is 30°. Find the breadth of the river.

5. A man on a river-bridge 50 feet high sees a barge at a depression of 30°. If the barge is travelling at the rate of 3 miles an hour, in how many seconds will it reach the bridge?

6. The string of a kite is inclined to the horizon at an angle of 60°. The length of string is 180 yards. Find the height of the kite.

7. In a room lighted by a single gas jet, the shadow of a vertical stick 1 foot long is observed to be 2 inches in length, the stick being a foot from directly under the jet. Find the height of the jet.

8. The top of a church spire has an elevation of 45°, the top of the tower, which is 100 feet high, an elevation of 30°. Find the height of the spire.

9. From a boat at sea the angles of elevation of the top and bottom of a tower 150 feet high on the top of a cliff are observed to be 60° and 45° respectively. Find the height of the cliff.

10. A person observes the angle subtended by a tower to be 30°, he walks $50\sqrt{3}$ feet towards its base and then finds that it subtends an angle of 60°. Find the height of the tower.

11. A man on the bank of a river observes that the elevation of a tree on the other side of the river directly opposite him is 60°, his eye being level with the ground. He then retires 100 feet and sees that the elevation is 45°. Find the breadth of the river and the height of the tree.

12. The angular distance between two posts on the bank of a river is found by a person standing on the other bank opposite their middle point to be 120°. He then goes back 200 yards and finds their angular distance to be 60°. Find the distance apart of the posts.

13. At the top of a house 50 feet high the elevation of a pillar is observed to be 45°, on the ground floor it is 60°. Find the height of the pillar.

14. Standing at the corner A of a field $ABCD$, a man observes that BC subtends an angle of 30°, walking from A to C, whose distance apart he finds to be 100 yards, he then observes that AB subtends an angle of 60°. Find the lengths of the sides of the field.

15. There are two posts which are 240 and 80 feet high respectively. From the foot of the second the elevation of the top of the first is found to be 60°. Find the elevation of the second from the foot of the first.

16. At a point midway between two posts they subtend angles of 30° and 60° respectively. Show that one of the posts is three times as high as the other.

17. A man six feet high stands midway between two telegraph posts which are 50 yards apart. The elevation of the top of each is 60°. Find their height.

18. A man in a balloon sees that two churches, which he knows to be a mile apart, subtend an angle of 60°. He is exactly above the middle point between them. Find his height.

19. Two observers 1000 yards apart, in the same vertical plane with a balloon but on opposite sides of it,

take its angles of elevation as 15° and 75°. Find the height of the balloon, having given that

$$\tan 15° = \cdot 27, \quad \tan 75° = 3\cdot 73.$$

20. A and B are two places on the sea shore, one mile apart, and C is an island at sea. The inclination of both AC and BC to AB is 60°. Find the distance of the island from the shore.

21. From one bank of a river the elevation is 30° of the top of a wall on the opposite bank whose height is known to be 56 feet. The height of the observer being 6 feet, find the width of the river.

22. From the summit of a tower whose height is 108 feet, the angles of depression of the top and bottom of a vertical column, standing on a level with the base of the tower, are found to be 30° and 60°; find the height of the column.

23. From the top of a ship's mast a buoy is seen to have a depression of 35°, on descending the mast a distance of 5 feet the depression is seen to be 34°. Find the distance of the buoy, having given that $\tan 35° = \cdot 7$, $\tan 34° = \cdot 675$.

24. A boy standing b feet behind, and opposite the middle of a football goal, sees that the elevation of the nearer cross-bar is α, and the elevation of the further one β.

Show that the length of the ground is $b(\tan\alpha\cot\beta - 1)$.

25. A valley is crossed by a horizontal bridge, of length l. The sides of the valley make angles α and β with the horizon; show that the height of the bridge above the bottom of the valley is

$$\frac{l}{\cot\alpha + \cot\beta}.$$

26. The height of a lighted candle is 6 inches, and the radius of its section is $\frac{3}{4}$ of an inch. The radius of the shadow cast by it is $5\frac{1}{4}$ inches. Find the height of the flame if the inclination to the horizon of the line joining the top of the flame to a point on the boundary of the shadow is α, having given that $\tan \alpha = \frac{4}{3}$.

27. Within a quadrangle the height of whose surrounding walls is half the side of its base, it is observed by a man 6 feet high that the elevation of one side of the quadrangle is 30° and that of the opposite side is 60°. Find his distance from these two sides.

28. The top ledge of a window subtends 60° at the eye of a person whose head is on a level with the bottom ledge of the window and distant b feet from it. The bottom ledge subtends 90°. Find the height and breadth of the window.

29. From the extremities of a ship 500 feet long, the angles which the direction of a buoy makes with that of the ship are 60° and 75°. Find the distance of the buoy from the ship; having given that

$$\cot 75° = 2 - \sqrt{3}.$$

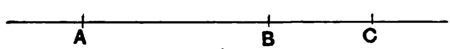
CHAPTER V.

THE SIGNS OF THE TRIGONOMETRICAL RATIOS OF ANY ANGLE.

31. Measurement of lines.

If it is required to measure from a given point a given length along a straight line supposed indefinitely prolonged in both directions, we must know in which *direction* the given length is to be measured off.

For the sake of clearness we agree that lines measured along the line from left to right shall be called positive, and consequently that lines measured from right to left shall be called negative.



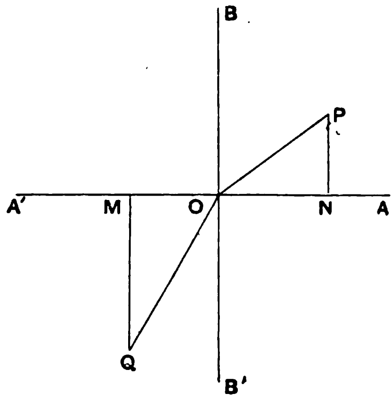
The length AB is thus positive, and BA negative, or $AB = -BA$.

If C be a third point anywhere on the straight line we have always $AB = AC + CB$; thus, if as in the figure, C is beyond B, the line CB is negative.

32. Take two lines AOA', BOB' at right angles.

Lines measured along or parallel to OA or OB are called

positive. Lines measured along or parallel to OA' or OB' are called *negative.*



Thus in the figure NP and ON are both positive, MQ and OM are both negative. A definite sign is thus always attached to NP and ON. This agreement as to the signs to be attached to lines, will be referred to subsequently as the *rule of signs.*

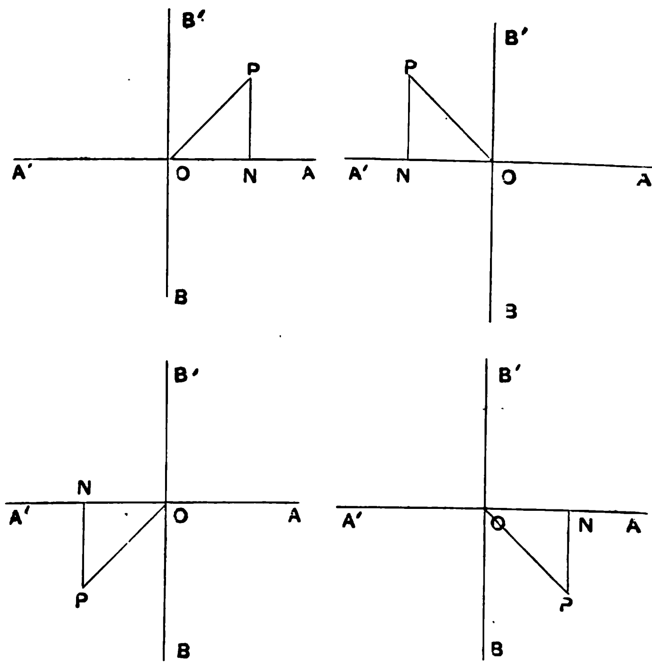
Observe that $PN = -NP$, and $QM = -MQ$. The four portions into which the indefinitely extended lines AOA', BOB' divide the plane are called *quadrants.*

33. Application of the rule of signs to the trigonometrical ratios.

By aid of the rule of signs, the definitions already given in Article 14, of the sine, cosine, &c., for an acute angle, may be extended so as to apply to angles of any magnitude.

From a point P on the bounding line let fall a perpendicular PN; then in whatever quadrant OP happens to be situated, the ratios of the last chapter are called the trigono-

metrical ratios of the angle AOP; namely, $\frac{NP}{OP}$ is $\sin AOP$, $\frac{ON}{OP}$ is $\cos AOP$, $\frac{NP}{ON}$ is $\tan AOP$; the reciprocals of these being the cosecant, secant, and cotangent of AOP.



From the rule of signs we see that NP is positive or negative according as P lies above or below AOA', and that ON is positive or negative according as P lies to the right or left of BOB'; *we take* OP *as being always positive.*

To show how the rule of signs affects the trigonometrical

ratios, take as an example the second figure. Here NP is positive and ON is negative, hence

$$\sin AOP = \frac{NP}{OP}, \text{ is positive;}$$

$$\cos AOP = \frac{ON}{OP}, \text{ is negative;}$$

$$\tan AOP = \frac{NP}{ON}, \text{ is negative.}$$

The proper signs of the ratios in each quadrant will be given below.

The cosecant, secant, and cotangent are, as before, the inverses of the sine, cosine, and cotangent, respectively. We thus again obtain equations (1) of Article 16.

It is seen, just as in Article 16, that

$$\tan A = \frac{\sin A}{\cos A}; \qquad \cot A = \frac{\cos A}{\sin A}.$$

Lastly from the fact that $OP^2 = NP^2 + ON^2$, it follows, as in Article 17, that

$$\sin^2 A + \cos^2 A = 1,$$

$$\sec^2 A = 1 + \tan^2 A,$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A.$$

Hence all the equations originally found in Chapter II., to connect the trigonometrical ratios of an acute angle, are now found to be applicable to the trigonometrical ratios of an angle of any magnitude.

34. Two important consequences of the definition of the trigonometrical ratios should be observed.

(i) The *arithmetical* values of the ratios obviously depend only on the acute angle OP makes with AOA', since that determines the shape of the triangle OPN.

Corresponding to any position of OP there are *three* others which give angles having their trigonometrical ratios arithmetically equal to those of the angle bounded by OP.

(ii) If from any given position the line OP turns through four right angles, or any multiple of four right angles, we get the set of angles (OA, OP). These angles have the same bounding lines, they therefore have the *same trigonometrical ratios.*

If the *arithmetically* smallest of the angles (OA, OP) be A, then this set of angles are all included in the expression $n \cdot 360° + A$, where n is any positive or negative integer, and A has its proper sign. See Article 3.

In circular measure, if α be the circular measure of A, the set of angles (OA, OP) are the following,

$$\alpha,\ 2\pi + \alpha,\ 4\pi + \alpha,\ 6\pi + \alpha \ldots \text{ and } -2\pi + \alpha,\ -4\pi + \alpha,\ -6\pi + \alpha \ldots$$

They are all included in $2n\pi + \alpha$; and we thus see that any trigonometrical ratio of $2n\pi + \alpha$ is equal to the same ratio of α.

Ex. Find sin 765°.

$$\begin{aligned} \sin 765° &= \sin (2 \cdot 360° + 45) \\ &= \sin 45°, \text{ by this Article,} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

35. To trace the changes of the sine of an angle as the angle increases from 0° to 360°.

Referring to the figures of Article 33, take OP as being a line of constant length and equal to r, calling $A\dot{O}P$ the angle A, since

$$\sin A = \frac{NP}{OP} = \frac{NP}{r},$$

we have merely to observe the changes of NP.

First Quadrant.

When $A=0$, NP is zero, hence $\sin 0^\circ = \frac{0}{r} = 0$.

As A increases so also does NP, till when $A=90^\circ$, we have $NP=r$, hence $\sin 90^\circ = \frac{r}{r} = 1$.

Thus we see that for the first quadrant $\sin A$ increases from zero to unity and is positive, since NP is positive.

Second Quadrant.

As A increases, NP diminishes; and when A is 180°, NP vanishes, therefore $\sin 180^\circ = \frac{0}{r} = 0$.

For the second quadrant $\sin A$ decreases from unity to zero, and is positive, since NP is positive.

Third Quadrant.

As A increases, NP increases (and becomes negative); when $A=270^\circ$ we have $NP=-r$, thus $\sin 270^\circ = \frac{-r}{r} = -1$.

For the third quadrant $\sin A$ changes from zero to -1, and is negative, since NP is negative.

Fourth Quadrant.

As A increases, NP diminishes (remaining negative); when A is 360° we have $NP=$ zero, thus $\sin 360^\circ = \frac{0}{r} = 0$.

For the fourth quadrant $\sin A$ changes from -1 to zero, and is negative, since NP is negative.

36. To trace the changes in the cosine of an angle.

In a precisely similar manner the changes of the cosine can be traced, the results being as follows:

$$\cos A = \frac{ON}{r}.$$

First Quadrant.

$$\cos 0^\circ = \frac{r}{r} = 1, \qquad \cos 90^\circ = \frac{0}{r} = 0.$$

Cos A decreases from unity to zero, and is positive.

Second Quadrant.

Cos A changes from zero to -1, and is negative.

Third Quadrant.

Cos A changes from -1 to zero, and is negative.

Fourth Quadrant.

Cos A changes from zero to unity, and is positive.

37. To trace the changes in the tangent of an angle.

$$\tan A = \frac{NP}{ON}.$$

First Quadrant.

In this quadrant NP and ON are both positive, as A increases, NP is increasing and ON diminishing; also

$$\tan 0^\circ = 0, \ \tan 90^\circ = \frac{r}{0} = \infty,$$

so that tan A increases from zero to infinity, and is positive.

Second Quadrant.

Here NP is positive and ON negative; NP decreases and ON increases, also $\tan 180^\circ = 0$.

For the second quadrant tan A is negative, and diminishes from infinity to zero.

Third Quadrant.

NP and ON are both negative; NP increases and ON diminishes; also $\tan 270 = \infty$.

For the third quadrant tan A is positive, and increases from zero to infinity.

The same values as in the first quadrant.

Fourth Quadrant.

NP is negative, and ON positive; NP diminishes and ON increases; also $\tan 360° = 0$.

For the fourth quadrant $\tan A$ is negative, and diminishes from infinity to zero.

38. Table of signs of trigonometrical ratios.

It is necessary to remember the signs of the trigonometrical ratios in the four quadrants; a useful table for that purpose is here given:

sine

+	+
−	−

cosine

−	+
−	+

tangent

−	+
+	−

The first figure states that $\sin A$ is positive if the bounding line of A lies in either of the first two quadrants, and negative if it lies in either of the last two.

An angle whose bounding line lies within any particular quadrant is called for brevity an angle *of that quadrant*.

We thus say that the second figure states the fact that angles of the first or fourth quadrants have the cosine positive, angles of the second or third have the cosine negative.

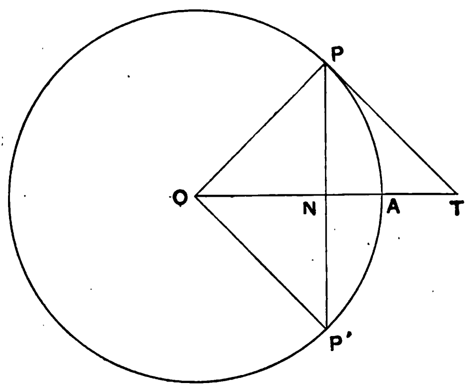
An inspection of the tables shows us that sines and cosines of angles of *opposite* quadrants have contrary signs, and that tangents of angles of *adjacent* quadrants have contrary signs. This fact will be subsequently referred to.

39. From Article 20, and the investigations of Articles 35–37, we see that the sine and cosine of an angle cannot be numerically greater than unity, and therefore that the secant and cosecant cannot be numerically less than unity.

The tangent and cotangent are seen to be capable of all values between $+\infty$ and $-\infty$.

40. When we take OP of invariable length r, P describes a circle.

If PN is produced to meet the circle at P', the $\angle POP'$ is double of the $\angle AOP$. Let the tangent at P to the circle meet OA in T.



The chord of the arc on which $\angle POP'$ stands is PP', or twice PN.

The sine, cosine, and tangent of the angle AOP are the ratios of NP, ON, PT, respectively, to OP.

Owing to this connexion of the trigonometrical ratios with a circle, they are sometimes called *circular functions* of the $\angle AOP$.

To say that any magnitude is a *function* of $\angle AOP$ expresses the fact that the magnitude changes only when $\angle AOP$ changes.

In the older treatises on Trigonometry, the lengths of the lines NP, ON, PT, and not their ratios to the radius, were defined to be the sine, cosine and tangent, respectively, of the angle AOP. With this definition, the sine, cosine, and tangent are functions not of the angle alone, but also of the radius of the circle.

CHAPTER VI.

Properties of the Trigonometrical Ratios.

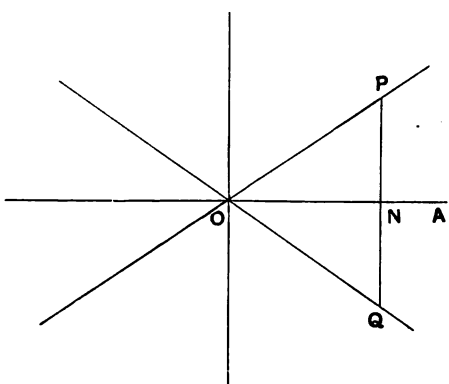
41. If two angles together make up 90°, each angle is said to be the *complement* of the other.

If two angles together make up 180°, each angle is said to be the *supplement* of the other.

For instance, the complement of 30° is 60°, but its supplement is 150°.

42. Connexion between the trigonometrical ratios of angles equal in magnitude but opposite in sign.

If the angle AOP be A, the angle AOQ will be $-A$,



where AOQ is arithmetically equal to AOP, but of opposite sign.

Take OP and OQ equal to each other, and join PQ cutting OA perpendicularly at N.

The triangles PON, QON have their sides equal in length, but NP and NQ are of opposite sign, hence

$$\sin(-A) = \frac{NQ}{OQ} = \frac{-NP}{OP} = -\sin A \quad\quad (1),$$

$$\cos(-A) = \frac{ON}{OQ} = \frac{ON}{OP} = \cos A \quad\quad (2).$$

From this it follows by division that

$$\tan(-A) = -\tan A \quad\quad (3).$$

43*. *To find all the angles having the same* cosine *as a given angle A.*

The set of angles (OA, OP) have the same cosine as A (Art. 34), similarly the set (OA, OQ) have the same cosine as $-A$, that is as A, by the last Article.

These two sets of angles are both included in the formula,

$$n \cdot 360° \pm A,$$

where n is a positive or negative integer.

Moreover these are the *only* angles having the same cosine as A; for there can be no other angle in either of the quadrants in which A and $-A$ are situated, and all angles in the quadrant opposite to that of A have their cosine opposite in sign to $\cos A$; similarly for all angles in the quadrant opposite to that of $-A$.

Expressed in circular measure the angles having the same cosine as a are

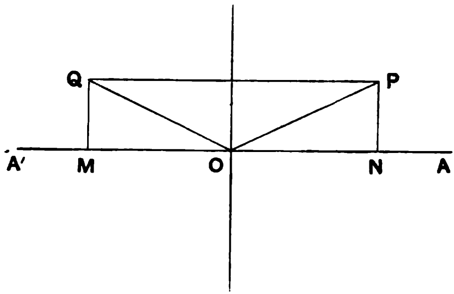
$$2n\pi \pm a,$$

where a is the circular measure of A.

These angles have also the same secant as A.

44. Connexion between the trigonometrical ratios of an angle and its supplement.

The angle AOP being A, the angle AOQ will be $180° - A$, the angles POA, QOA' being equal.



Take OQ equal to OP, and draw PN, QM perpendicular to AOA'. The sides of the triangles PON, QOM are equal in length, but ON and OM have contrary signs; hence

$$\sin AOQ = \frac{MQ}{OQ} = \frac{NP}{OP} = \sin AOP,$$

or $$\sin(180° - A) = \sin A \quad \text{.....................(4)},$$

and $$\cos AOQ = \frac{OM}{OQ} = \frac{-ON}{OP} = -\cos A,$$

or $$\cos(180° - A) = -\cos A \quad \text{..................(5)}.$$

From this it follows that

$$\tan(180° - A) = -\tan A \quad \text{..................(6)}.$$

45*. *To find all the angles having the same* sine *as a given angle A.*

The angles (OA, OP) have the same sine as A; the angles (OA, OQ) have the same sine as $180° - A$, that is as A, by the last Article.

These two sets are respectively

$$n \,.\, 360° + A, \quad n \,.\, 360° + 180° - A,$$

or in circular measure

$$2n\pi + \alpha, \quad (2n+1)\,\pi - \alpha,$$

where n is any positive or negative integer, and α the circular measure of A.

Both formulæ happen to be included in the one expression

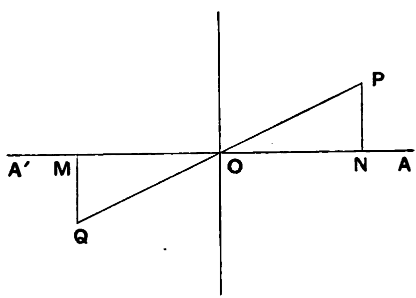
$$m\pi + (-1)^m \alpha,$$

where m is any positive or negative integer.

These angles have also the same cosecant as A.

46. Connexion between the trigonometrical ratios of angles differing by 180°.

Produce OP bounding the angle A to Q, making OQ equal to OP.



The angle AOQ is $180° + A$. Let fall the perpendiculars PN and QM.

The triangles PON, QOM have their hypotenuses equal, and their sides equal and of contrary sign, hence

$$\sin AOQ = \frac{MQ}{OQ} = \frac{-NP}{OP} = -\sin AOP,$$

or $$\sin(180° + A) = -\sin A \quad\text{.....................(7)},$$

and $$\cos AOQ = \frac{OM}{OQ} = \frac{-ON}{OP} = -\cos AOP,$$

or $$\cos(180° + A) = -\cos A \quad\text{.....................(8)}.$$

From this it follows that

$$\tan(180° + A) = \tan A \quad\text{.......................(9)}.$$

47*. *To find all the angles whose* tangent *is the same as that of a given angle A.*

We see from the last Article, that (OA, OP) and (OA, OQ) have the same tangent as A. And these are the only angles having the same tangent as A; for the two quadrants to which they do not belong are both *adjacent* to that of A and therefore, by Article 38, give angles whose tangents have the opposite sign to $\tan A$.

Hence the required angles are all included in

$$n \,.\, 360° + A, \quad \text{or} \quad n \,.\, 360° + 180° + A.$$

In circular measure they are $2n\pi + \alpha$; $2n\pi + \pi + \alpha$. These are both included in

$$m\pi + \alpha,$$

where m is any positive or negative integer, and α the circular measure of A.

These angles have also the same cotangent as A.

48. Connexion between the trigonometrical ratios of angles whose sum or whose difference is 90°.

Draw OQ perpendicular to OP, making OQ equal to OP; and let fall perpendiculars PN, QM.

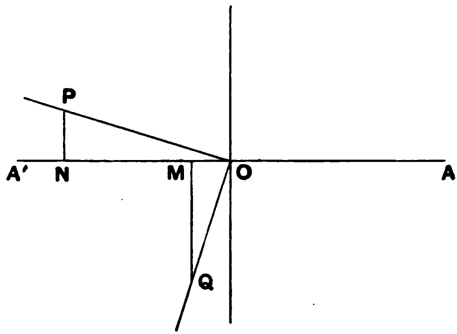
The triangles PON, QOM are seen to be equal, also

$$NP = -OM, \qquad MQ = ON.$$

In the figure both MQ and ON are negative.

Hence, $$\sin AOQ = \frac{MQ}{OQ} = \frac{ON}{OP} = \cos AOP,$$

$$\cos AOQ = \frac{OM}{OQ} = \frac{-NP}{OP} = -\sin AOP.$$



First, let A be the angle AOP, then $\angle AOQ = 90° + A$, and the above equations give us,

$$\sin (90° + A) = \cos A \text{ .... } (10),$$

$$\cos (90° + A) = -\sin A \text{ .... } (11).$$

From which it follows that

$$\tan (90° + A) = -\cot A \text{ .... } (12).$$

Next, let AOQ be taken equal to A, then

$$\angle AOP = A - 90°;$$

thus $\sin A = \cos (A - 90°)$, or by Article 42,

$$\sin A = \cos (90° - A) \text{ .... } (13),$$

also $\cos A = -\sin (A - 90°)$, or by Article 42,

$$\cos A = \sin (90° - A) \text{ .... } (14).$$

From which it follows that

$$\tan A = \cot (90° - A) \text{ .... } (15).$$

The last three results have already been given for the case where A is an acute angle, see Art. 19.

49. The following table contains the results of the preceding equations from (1) to (15).

	$-A$	$180-A$	$180+A$	$90+A$	$90-A$
sine	$-\sin A$	$\sin A$	$-\sin A$	$\cos A$	$\cos A$
cosine	$\cos A$	$-\cos A$	$-\cos A$	$-\sin A$	$\sin A$
tangent	$-\tan A$	$-\tan A$	$\tan A$	$-\cot A$	$\cot A$

The results of the table may be applied to find other results of the same kind, as follows:

Ex. 1. Find $\sin(270° - B)$.

$$\sin(270° - B) = \sin(90° + 180° - B)$$
$$= \cos(180° - B), \text{ by (10)},$$
$$A \text{ being replaced by } 180° - B,$$
$$= -\cos B, \text{ by (5)}.$$

Or thus;

$$\sin(270 - B) = \sin(180° + 90° - B)$$
$$= -\sin(90° - B), \text{ by (7)},$$
$$A \text{ being replaced by } 90° - B,$$
$$= -\cos B, \text{ by (14)}.$$

Ex. 2. Find $\cos(540° + C)$.

$$\cos(540° + C) = \cos(360° + 180° + C)$$
$$= \cos(180° + C), \text{ by Article 34, (ii.)}$$
$$= -\cos C, \text{ by (8)}.$$

Ex. 3. Find $\tan 330°$.

$$\begin{aligned}\tan 330° &= \tan(180° + 90° + 60°)\\ &= \tan(90° + 60), \text{ by (9)},\\ &= -\cot 60°, \text{ by (12)},\\ &= -\frac{1}{\sqrt{3}}.\end{aligned}$$

Or thus;

$$\begin{aligned}\tan 330° = \tan(360° - 30) &= \tan(-30°), \text{ Article 34},\\ &= -\tan 30°, \text{ by (3)},\\ &= -\frac{1}{\sqrt{3}}.\end{aligned}$$

The importance of the results contained in the table makes the following rule a useful one; it applies to all cases, and includes those in the table.

(1) Any trigonometrical ratio of (even multiple of $90° \pm A$) is equal *numerically* to the *same trigonometrical ratio* of A.

(2) Any trigonometrical ratio of (odd multiple of $90° \pm A$) is equal *numerically to the corresponding co-ratio* of A.

How to determine the *sign* will best be seen from an example.

Find $\sin(270° + A)$; here we have an odd multiple of $90° + A$, we therefore take the co-ratio of $\sin A$ which is $\cos A$, hence, $\sin(270° + A)$ is equal numerically to $\cos A$, and since $270° + A$ is of the 4th quadrant, supposing A to be acute, its sine is *negative*;

$$\therefore \sin(270° + A) = -\cos A.$$

Find $\tan(540° - A)$; we have an even multiple of $90° - A$, we therefore take the *same trigonometrical ratio* of A, thus

$\tan(540° - A)$ is equal numerically to $\tan A$.

And since $540° - A$ is of the second quadrant supposing A to be acute, its tangent is *negative*;

$$\therefore \tan(540° - A) = -\tan A.$$

50. In the figures of Articles 42, 44, and 46, we have taken A as being an acute angle. The results of those Articles are, however, true, whatever may be the value of A. It would be a good exercise to draw the figures in the case when A is greater than 90°. The proof will be found to apply identically.

EXAMPLES. XII.

1. Use the results of equations (1)—(15) to find the values of the following expressions.

(1) $\sin 120°$.

$$\left[\sin 120° = \sin(180° - 60°) = \sin 60°, \text{ by (4)}, \; = \frac{\sqrt{3}}{2}\right].$$

(2) $\cos 120°$. (3) $\tan 120°$. (4) $\cot 150°$.

(5) $\sec 225°$. (6) $\sin 240°$. (7) $\tan 210°$.

(8) $\cos 135°$.

2. Find the values of

(1) $\tan 300°$. (2) $\cot 315°$. (3) $\operatorname{cosec} 330°$.

(4) $\sin 1125°$. (5) $\cos 900°$.

3. Show by using equation (13) that $\sin 45° = \cos 45°$.

4. Find $\operatorname{cosec}\left(2\pi + \frac{5\pi}{6}\right)$.

5. Apply the rule given in Article 49 to find the values of

(1) $\cos(270° - A)$. (2) $\sin(360° - A)$.

(3) $\cot(360° - A)$. (4) $\tan(270° + A)$.

(5) $\sin(360° + A)$. (6) $\cos(720° - A)$.

(7) $\cot(720° + A)$. (8) $\operatorname{cosec} 420°$.

6. By writing $90° + B$ for A in equations (4), (5), and (6), deduce equations (13), (14) and (15), making use of (10), (11) and (12).

7. By writing $90° - B$ for A in equations (10), (11) and (12), deduce equations (4), (5) and (6), making use of (13), (14), (15).

8. From the proof of Article 48, if A be the angle AOQ, show that

$$\sin A = -\sin(A - 180°).$$

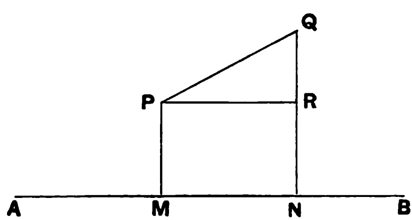
9. Draw the figure when the angle A is *obtuse*, in equations (1), (2) and (3). (The proof will then apply identically.)

CHAPTER VII.

THE TRIGONOMETRICAL RATIOS OF TWO OR MORE ANGLES.

51. The projections of lines.

IF from the extremities PQ of any line PQ, perpendiculars PM, QN be drawn to any line AB, the portion MN intercepted is called the *projection* of PQ on AB.



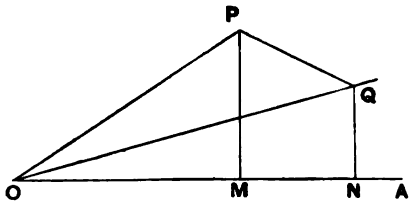
Through P draw PR parallel to AB, then if QPR be A, we have

$$MN = PR = PQ \cos A.$$

Thus the projection of a line on another line AB is equal to the first line multiplied by the cosine of its inclination to AB.

52. Projections of the sides of a triangle.

From the preceding definition of projection there follows simple connexion between the projections of the sides of triangle.



From the vertices P and Q of the triangle OPQ draw PM and QN perpendicular to OA; it is obvious that

$$ON = OM + MN,$$

$$MN = ON - OM,$$

or in other words

projection of OQ = projection of OP + projection of PQ,

projection of PQ = projection of OQ − projection of OP.

53. Trigonometrical ratios of the sum of two angles.

To show that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

If the angles AOC and COD (see next page) be called A and B respectively, the angle AOD is equal to $A+B$.

In OD the line bounding the angle $A+B$ take any point P, and from P draw PK perpendicular to OC.

Project the sides of the triangle POK on the line OA, then as in the last Article

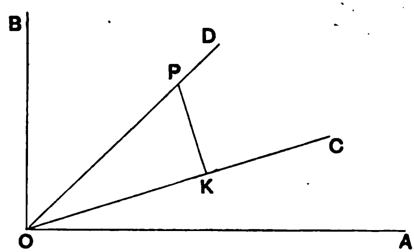
projection of OP = projection of OK – projection of PK,

and the angle between $\begin{cases} OP \text{ and } OA \text{ is } A+B, \\ OK \text{ and } OA \text{ is } A, \\ PK \text{ and } OA \text{ is } 90^\circ - A, \end{cases}$

therefore, by Article 52,

$$OP \cos(A+B) = OK \cos A - PK \cos(90^\circ - A),$$

but $\quad OK = OP \cos B,$ and $PK = OP \sin B,$



hence

$$OP \cos(A+B) = OP \cos B \cos A - OP \sin B \cos(90^\circ - A),$$

writing $\sin A$ for $\cos(90^\circ - A)$, by Article 48, and omitting OP, we have

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \ldots\ldots\ldots(a).$$

Next project on OB

projection of OP = projection of OK + projection of KP,

and the angle between

$$\begin{cases} OP \text{ and } OB \text{ is } 90^\circ - (A+B), \\ OK \text{ and } OB \text{ is } 90^\circ - A, \\ KP \text{ and } OB \text{ is } A; \end{cases}$$

$$\therefore \; OP \cos\{90^\circ - (A+B)\} = OK \cos(90^\circ - A) + KP \cos A,$$

hence substituting the values of OK and KP as before,

$$OP\cos\{90^\circ-(A+B)\}=OP\cos B\cos(90^\circ-A)+OP\sin B\cos A;$$

omitting OP, and using Article 48, we have

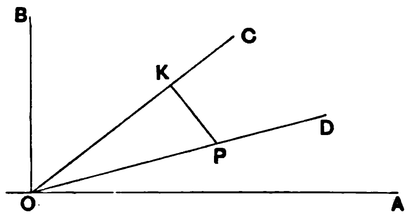
$$\sin(A+B)=\sin A\cos B+\cos A\sin B\ldots\ldots\ldots(b).$$

54. Trigonometrical ratios of the difference of two angles.

To show that

$$\mathbf{\sin(A-B)=\sin A\cos B-\cos A\sin B}$$
$$\mathbf{\cos(A-B)=\cos A\cos B+\sin A\sin B.}$$

The angles AOC and DOC being A and B, the angle AOD is equal to $A-B$.



In OD, the line bounding $A-B$, take any point P, from P draw PK perpendicular to OC.

Projecting on OA, we have

projection of OP = projection of OK + projection of KP,

$$\text{and the angle between }\begin{cases} OP \text{ and } OA \text{ is } A-B,\\ OK \text{ and } OA \text{ is } A,\\ KP \text{ and } OA \text{ is } 90^\circ-A.\end{cases}$$

Hence,

$$\begin{aligned}OP\cos(A-B)&=OK\cos A+KP\cos(90^\circ-A)\\ &=OP\cos B\cos A+OP\sin B\sin A,\end{aligned}$$

or $$\cos(A-B)=\cos A\cos B+\sin A\sin B\ldots\ldots(c).$$

Again, projecting on OB, we have

$$OP\cos\{90^\circ-(A-B)\}=OK\cos(90^\circ-A)-KP\cos A$$
$$=OP\cos B\sin A-OP\sin B\cos A,$$

thus $$\sin(A-B)=\sin A\cos B-\cos A\sin B\ldots\ldots(d).$$

The formulæ (a), (b), (c) and (d) are of the greatest importance, and should be carefully remembered. We notice that (c) and (d) are obtained from (a) and (b) by writing in them $-B$ in place of B. Observe that these results are only proved true when $A+B$ is an acute angle, but an extended proof is given at the end of the chapter, so that we shall henceforth assume them to be true for angles of any magnitude.

The following examples illustrate the above results.

Ex. 1. Find the value of $\sin(A+180^\circ)$;
by (a),

$$\sin(A+180^\circ)=\sin A\cos 180^\circ+\cos A\sin 180^\circ$$
$$=-\sin A;$$

since $$\sin 180^\circ=0,\quad \cos 180^\circ=-1.$$

Ex. 2. Find $\sin 75^\circ$;

$$\sin 75^\circ=\sin(45^\circ+30^\circ)$$
$$=\sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ$$
$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Ex. 3. Given $\sin A=\frac{1}{3}$, $\cos B=\frac{1}{8}$, find $\cos(A+B)$; A and B being of the first quadrant.

Since,

$$\sin A=\tfrac{1}{3},\quad \cos A=\sqrt{1-\tfrac{1}{9}}=\frac{2\sqrt{2}}{3}.$$

and $$\sin B=\sqrt{1-\tfrac{1}{64}}=\frac{3\sqrt{7}}{8}.$$

Hence

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$= \frac{2\sqrt{2}}{3} \cdot \frac{1}{8} - \frac{1}{3} \cdot \frac{3\sqrt{7}}{8} = \frac{1}{6\sqrt{2}} - \frac{\sqrt{7}}{8}.$$

EXAMPLES. XIII.

1. Show that $\cos 75^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

2. Prove that $\sin(45^\circ + A) = \dfrac{\cos A + \sin A}{\sqrt{2}}$.

3. Prove that $\sin(45^\circ - A) = \cos(A + 45^\circ)$.

4. Find $\sin 15^\circ$ and $\cos 15^\circ$.

$$\left[\sin 15^\circ = \sin(45^\circ - 30^\circ) = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}\right].$$

5. Show that $\operatorname{cosec}(A + 60^\circ) = \dfrac{2}{\sin A + \sqrt{3}\cos A}$.

6. Prove that $\sin(30^\circ - \alpha) + \sin(30^\circ + \alpha) = \cos\alpha$.

$$\cos\left(30^\circ - \frac{\beta}{2}\right) - \cos\left(30^\circ + \frac{\beta}{2}\right) = \sin\frac{\beta}{2}.$$

7. Show that $\tan A + \tan B = \dfrac{\sin(A+B)}{\cos A \cos B}$.

8. Show that $\cot B - \cot A = \dfrac{\sin(A-B)}{\sin A \sin B}$.

9. Show that

$$(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 2 + 2\cos(A - B).$$

10. Show that

$$(\sin A + \cos B)^2 + (\sin B + \cos A)^2 = 2 + 2\sin(A + B).$$

11. If $\sin A = \frac{11}{61}$, $\sin B = \frac{9}{41}$, find the value of

$$\cos(A - B).$$

12. If $\cos A = \frac{3}{5}$, $\operatorname{cosec} B = \frac{13}{5}$, find $\sin (A - B)$.

13. If $\sin A = \frac{8}{17}$, $\cos B = \frac{60}{61}$, find $\cos (A + B)$.

14. If $\sec A = 113$, $\operatorname{cosec} B = \frac{45}{27}$, find $\sin (A + B)$.

15. If $\tan A = \dfrac{1}{2\sqrt{2}}$, $\tan B = \frac{3}{4}$, find $\sec (A - B)$.

16. If $\tan A = a$, $\tan B = b$,

show that $$\sin (A + B) = \frac{a + b}{\sqrt{(1 + a^2)(1 + b^2)}}.$$

17. Prove that

$$\sin (\alpha + \beta) + \cos (\alpha - \beta) = (\sin \alpha + \cos \alpha)(\sin \beta + \cos \beta).$$

18. Prove that

$$\sin (A + 2B) \cos A + \cos (A + 2B) \sin A = \sin (2A + 2B).$$

19. Prove that

$$\sin A \cos (B + C) - \sin B \cos (A + C) = \sin (A - B) \cos C.$$

20. Prove that $\sin (\alpha + \beta) \sin (\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$.

21. Prove that

$$\cos (A + B) \cos (A - B) = 1 - \sin^2 A - \sin^2 B.$$

22. Prove that $\sin 3A \cos 2A + \cos 3A \sin 2A = \sin 5A$.

23. Prove that

$$\sin (A + B) \cos B - \cos (A + B) \sin B = \sin A.$$

24. Prove that $\cos (\alpha + \beta) \cos \beta + \sin (\alpha + \beta) \sin \beta = \cos \alpha$.

25. Prove that $\cos \theta \cos 3\theta + \sin \theta \sin 3\theta = \cos 2\theta$.

26. Prove that

$$(\cos A - \sin A \sin 2A)^2 + (\sin A + \cos 2A \sin A)^2 = \cos^2 A.$$

27. Prove that $1 + \tan A \tan 2A = \sec 2A$.

28. Prove that $\sin^2 (A + 45^\circ) + \sin^2 (A - 45^\circ) = 1$.

55. To prove that

$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A.$$

If we take B equal to A in the addition formulæ (a) and (b) they become

$$\sin 2A = 2 \sin A \cos A \qquad \text{(i)}$$
$$\cos 2A = \cos^2 A - \sin^2 A.$$

If in the last expression we substitute $1 - \cos^2 A$ for $\sin^2 A$, it becomes

$$\cos 2A = 2 \cos^2 A - 1,$$

and again by substituting for $\cos^2 A$ its value $1 - \sin^2 A$, we have

$$\cos 2A = 1 - 2 \sin^2 A,$$

hence

$$\left.\begin{array}{l} \cos^2 A - \sin^2 A \\ 2 \cos^2 A - 1 \\ 1 - 2 \sin^2 A \end{array}\right\} = \cos 2A \qquad \text{(ii)}.$$

Also

$$\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{2 \sin A \cos A}{\cos^2 A - \sin^2 A},$$

and dividing the numerator and the denominator of the last expression by $\cos^2 A$, we have,

$$\tan 2A = \frac{\dfrac{2 \sin A}{\cos A}}{1 - \dfrac{\sin^2 A}{\cos^2 A}},$$

or,

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \qquad \text{(iii)}.$$

The formulæ (i), (ii), and (iii) should be committed to memory.

Two other formulæ are sometimes useful.

(a)
$$\frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{1 - \dfrac{\sin^2 A}{\cos^2 A}}{1 + \dfrac{\sin^2 A}{\cos^2 A}} = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \cos 2A;$$

thus $$\frac{1-\tan^2 A}{1+\tan^2 A} = \cos 2A.$$

(β) $$\tan A = \frac{\sin A}{\cos A} = \frac{2\sin A\cos A}{2\cos^2 A} = \frac{\sin 2A}{1+\cos 2A},$$

or, $$\tan A = \frac{\sin 2A}{1+\cos 2A}.$$

Direct geometrical proofs will be given of most of these theorems in the following chapter.

Ex. 1. If $\cos A = \frac{1}{3}$, find $\cos 2A$.

$$\cos 2A = 2\cos^2 A - 1$$

by (ii), of this Article

$$= \tfrac{2}{9} - 1 = -\tfrac{7}{9}.$$

Ex. 2. If $\cos A = \frac{1}{5}$, find $\sin\frac{A}{2}$.

$$2\sin^2\frac{A}{2} = 1 - \cos A$$

by (ii) of this Article

$$= 1 - \tfrac{1}{5}, \text{ or } \sin\frac{A}{2} = \sqrt{\tfrac{2}{5}}.$$

Ex. 3. Show that

$$\cos^2 A + \sin^2 A\cos 2B = \cos^2 B + \sin^2 B\cos 2A.$$

Multiply both expressions by 2, and replace

$$2\cos^2 A \text{ by } 1+\cos 2A,$$
$$2\cos^2 B \text{ by } 1+\cos 2B;$$

the left side becomes

$$1+\cos 2A + (1-\cos 2A)\cos 2B$$
$$= 1+\cos 2A+\cos 2B-\cos 2A\cos 2B,$$

the right side becomes

$$1+\cos 2B + (1-\cos 2B)\cos 2A$$
$$= 1+\cos 2A+\cos 2B-\cos 2A\cos 2B.$$

Thus the expressions are seen to be equal.

The above method of introducing the cosine of the double angle instead of the square of the sine or cosine of the angle, is of frequent use.

EXAMPLES. XIV.

1. Given the values of sin 30° and cos 30°, deduce those of sin 60° and cos 60°.

2. Given $\cos 30^\circ = \frac{\sqrt{3}}{2}$, prove that $\cos 15^\circ = \frac{\sqrt{2+\sqrt{3}}}{2}$.

3. If $\sin A = \frac{1}{5}$, find $\cos 2A$ and $\sin 2A$.

4. If $\cos A = \frac{3}{5}$, find $\tan 2A$.

5. If $\cos 2A = \frac{13}{36}$, show that $\cos A = \frac{7}{6\sqrt{2}}$.

6. Given that $\sin A = \frac{1}{4}$, show that $\sin 2A = \frac{\sqrt{15}}{8}$.

7. Show that $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$.

8. From (β) show that $\tan 22\frac{1}{2}^\circ = \frac{1}{\sqrt{2}+1}$.

9. Prove that $\frac{1 - \cos 2A}{\sin 2A} = \tan A$.

10. Prove that $2 \operatorname{cosec} 2A = \sec A \,.\, \operatorname{cosec} A$.

11. Prove that $(\sin A + \cos A)^2 = 1 + \sin 2A$.

12. Prove that $(\sin A - \cos A)^2 = 1 - \sin 2A$.

13. Prove that $\cos 2\alpha \,.\, \sec^2 \alpha = 1 - \tan^2 \alpha$.

14. Prove that $\tan A + \cot A = 2 \operatorname{cosec} 2A$.

15. Prove that $\cot A - \tan A = 2 \cos 2A \operatorname{cosec} 2A$.

16. Prove that $\cos^4 A - \sin^4 A = \cos 2A$.

17. Prove that $\dfrac{1-\cos 2A}{1+\cos 2A}=\tan^2 A$; and hence that

$$\tan 112\tfrac{1}{2}^\circ=\frac{1}{1-\sqrt{2}}.$$

18. Prove that $\cos\left(\dfrac{\pi}{4}-\theta\right)\cos\left(\dfrac{\pi}{4}+\theta\right)=\frac{1}{2}\cos 2\theta.$

19. Prove that $\dfrac{\operatorname{cosec} 2a}{1+\operatorname{cosec} 2a}=\dfrac{1+\tan^2 a}{(1+\tan a)^2}.$

20. Prove that $\operatorname{cosec} 2\theta+\cot 2\theta=\cot\theta.$

21. Prove that $\sin^2(A+B)-\sin^2(A-B)=\sin 2A \,.\, \sin 2B.$

22. Prove that $2\sin^2(45^\circ-A)=1-\sin 2A.$

23. Prove that

$$\cos^2(A+15^\circ)+\cos^2(A-15^\circ)=1+\frac{\sqrt{3}}{2}\cos 2A.$$

24. Prove that

$$\cos^2 A+\cos^2(A+60^\circ)+\cos^2(A-60^\circ)=\tfrac{3}{2}.$$

25. Prove that $\dfrac{1-\tan^2\left(\dfrac{\pi}{4}-a\right)}{1+\tan^2\left(\dfrac{\pi}{4}-a\right)}=\sin 2a.$

26. Prove that $\dfrac{2\sin A+\sin 2A}{2\sin A-\sin 2A}=\cot^2\dfrac{A}{2}.$

27. Show that $\cos 18^\circ=\sin 72^\circ.$

Hence prove that

$$1=4\sin 18^\circ(1-2\sin^2 18^\circ).$$

56. Summary of results; new formulæ.

In Articles 53 and 54, the following important equations have been proved:

$$\sin(A+B)=\sin A\cos B+\cos A\sin B,$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B,$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B,$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

From these, by addition and subtraction we get

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B,$$
$$\sin(A+B)-\sin(A-B)=2\cos A\sin B,$$
$$\cos(A+B)+\cos(A-B)=2\cos A\cos B,$$
$$\cos(A-B)-\cos(A+B)=2\sin A\sin B.$$

Let $A+B=C$, $A-B=D$, then since $A=\frac{C+D}{2}$, $B=\frac{C-D}{2}$, we get from the last set of formulæ, the following new ones,

$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\;\ldots\ldots\ldots(1),$$

$$\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\;\ldots\ldots\ldots(2),$$

$$\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\;\ldots\ldots\ldots(3),$$

$$\cos D-\cos C=2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\;\ldots\ldots\ldots(4).$$

On account of their importance it is usual to remember the last four results in words, as follows:

The sum of two sines = twice sine of half the sum multiplied by cosine of half the difference.

The difference of two sines = twice cosine of half the sum multiplied by sine of half the difference.

The sum of two cosines = twice cosine of half the sum multiplied by cosine of half the difference.

The difference of two cosines = twice sine of half the sum multiplied sine of half the difference reversed.

Ex. 1. Express $\sin A+\cos B$ as a product.

$$\sin A+\cos B=\sin A+\sin\left(\frac{\pi}{2}-B\right),$$

or by (1),

$$= 2 \sin\left(\frac{\pi}{4} + \frac{A - B}{2}\right) \cos\left(\frac{\pi}{4} - \frac{A + B}{2}\right).$$

Or we might write $\cos\left(\frac{\pi}{2} - A\right)$ for $\sin A$, thus getting the sum of two cosines, then use (3). Prove that we get the same result.

For instance,

$$\sin 25^\circ + \cos 15^\circ = 2 \sin\left(45^\circ + \frac{25^\circ - 15^\circ}{2}\right) \cos\left(45^\circ - \frac{25^\circ + 15^\circ}{2}\right)$$

$$= 2 \sin 50^\circ . \cos 25^\circ.$$

Ex. 2. Simplify $\dfrac{\sin A - \sin B}{\sin A + \sin B}$.

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2 \sin \frac{A - B}{2} \cos \frac{A + B}{2}}{2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}} = \tan \frac{A - B}{2} \cot \frac{A + B}{2}.$$

Ex. 3. Simplify $\sin A + \sin 2A + \sin 3A$.

Take together the first and last terms thus,

$$\sin A + \sin 3A = 2 \sin \frac{3A + A}{2} \cos \frac{3A - A}{2} = 2 \sin 2A \cos A$$

and $\sin 2A = 2 \sin A \cos A$,

hence

$$\sin A + \sin 2A + \sin 3A = 2 \cos A (\sin 2A + \sin A)$$

$$= 2 \cos A . 2 \sin \frac{3A}{2} \cos \frac{A}{2}$$

$$= 4 \cos \frac{A}{2} \cos A \sin \frac{3A}{2}.$$

EXAMPLES. XV.

Prove the following results:

1. $\sin 60^\circ + \sin 40^\circ = 2 \sin 50^\circ \cos 10^\circ$.

2. $\sin 70^\circ - \sin 50^\circ = 2 \cos 60^\circ \sin 10^\circ$.

3. $\cos 73^\circ + \cos 81^\circ = 2 \cos 77^\circ \cos 4^\circ$.

4. $\cos 1^\circ - \cos 11^\circ = 2 \sin 6^\circ \sin 5^\circ$.

5. Express as products

   (i) $\sin 70^\circ + \sin 30^\circ$. (ii) $\sin 30^\circ - \sin 15^\circ$.

   (iii) $\cos 17^\circ - \cos 77^\circ$. (iv) $\cos 1^\circ + \cos 3^\circ$.

Show that the following equations are true:

6. $\sin 80^\circ + \sin 40^\circ = \sqrt{3} \cos 20^\circ$.

7. $\cos 45^\circ + \cos 75^\circ = \cos 15^\circ$.

8. $\sin (A + 60^\circ) - \sin (A - 60^\circ) = \sqrt{3} \cos A$.

9. $\sin(2A+4B)+\sin(4A+2B)=2\sin(3A+3B)\cos(A-B)$.

10. $\sin 2A + \sin 8A = 2 \sin 5A \cos 3A$.

11. $\cos 3A + \cos 5A = 2 \cos A \cos 4A$.

12. $\sin (A + B - C) - \sin (A - B + C) = 2 \cos A \sin (B - C)$.

13. $\cos A - \cos 15A = 2 \sin 7A \sin 8A$.

14. $\cos (n-1) A + \cos (n+1) A = 2 \cos A \cos nA$.

15. $\{\sin (A - B) + \sin (A + 3B)\} \sec 2B$
$$= (\cos 2B - \cos 2A) \operatorname{cosec} (A - B).$$

16. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan \dfrac{A+B}{2}$. [Notice that $\cos \dfrac{A-B}{2}$ divides out.]

17. $\dfrac{\sin \alpha - \sin \beta}{\cos \alpha + \cos \beta} = \tan \dfrac{\alpha - \beta}{2}$.

18. $\dfrac{\sin\theta - \sin\phi}{\cos\phi - \cos\theta} = \cot\dfrac{\theta+\phi}{2}$.

19. $\dfrac{\sin a + \sin 3a}{\cos a + \cos 3a} = \tan 2a$.

20. $\dfrac{\sin 80^\circ + \sin 10^\circ}{\cos 80^\circ + \cos 10^\circ} = \cot 45^\circ$.

21. $\dfrac{\cos 50^\circ - \cos 70^\circ}{\sin 70^\circ - \sin 50^\circ} = \sqrt{3}$.

22. $\dfrac{\sin\dfrac{5B}{2} - \sin\dfrac{3B}{2}}{\cos\dfrac{3B}{2} - \cos\dfrac{5B}{2}} = \cot 2B$.

23. $\dfrac{\cos\dfrac{7A}{12} + \cos\dfrac{A}{3}}{\sin\dfrac{7A}{12} - \sin\dfrac{A}{3}} = \cot\dfrac{A}{8}$.

24. $\dfrac{\sin(A+3B) + \sin(3A+B)}{\sin 2A + \sin 2B} = 2\cos(A+B)$.

25. $\cos 25^\circ - \sin 5^\circ = \cos 35^\circ$.

26. $\sin\dfrac{2\pi}{7} + \sin\dfrac{4\pi}{7} - \sin\dfrac{6\pi}{7} = 4\sin\dfrac{\pi}{7}\sin\dfrac{3\pi}{7}\sin\dfrac{5\pi}{7}$.

27. If $\dfrac{\cos(A+B) - \cos(B+C)}{\cos(A-B) - \cos(B-C)} = \dfrac{\cos(B+C) - \cos(C+A)}{\cos(B-C) - \cos(C-A)}$,

then will $\dfrac{\tan B}{\tan\dfrac{C+A}{2}} = \dfrac{\tan C}{\tan\dfrac{A+B}{2}}$.

28. $(\sin\theta + \cos\theta)(\sin 2\theta + \cos 2\theta) = \cos\theta - \cos\left(3\theta + \dfrac{\pi}{2}\right)$.

29. $\cos 60^\circ\, 1' = \cos 1' - \cos 59^\circ\, 59'$.

30. $\cos 60^\circ + 2\cos 70^\circ + \cos 80^\circ = 4\cos^2 5^\circ \cdot \cos 70^\circ$.

31. $\cos A - \cos B - \sin(A-B)$

$$= 2\sin\frac{B-A}{2}\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)\left(\sin\frac{B}{2}+\cos\frac{B}{2}\right).$$

32. $\dfrac{\sin 45^\circ + \cos 75^\circ}{\sin 45^\circ - \cos 75^\circ} = \dfrac{1}{\sqrt{3}}\cot 15^\circ.$

57. It is often required to change products of sines and cosines into a sum. For this purpose the formulae at the top of page 85 are needed. They may be expressed in words as follows:

the first two;

twice the product of sine and cosine

= sine of sum + sine difference of angles:

the other two;

twice the product of two cosines

= cosine of sum + cosine of difference of angles:

twice the product of two sines

= cosine of difference − cosine of sum of two angles.

Ex. 1. Express as a sum $2\sin 15^\circ \cos 10^\circ$.

Ans. $\sin 25^\circ + \sin 5^\circ$.

Ex. 2. Express as a sum $2\sin(A+B)\cos(C+D)$, we get, by the first rule,

$$\sin(A+B+C+D)+\sin(A+B-C-D).$$

Ex. 3. Show that

$$\sin A\sin(B-C)+\sin B\sin(C-A)+\sin C\sin(A-B)=0,$$

we have

$$\sin A\sin(B-C)=\tfrac{1}{2}\{\cos(A-B+C)-\cos(A+B-C)\},$$
$$\sin B\sin(C-A)=\tfrac{1}{2}\{\cos(B-C+A)-\cos(B+C-A)\},$$
$$\sin C\sin(A-B)=\tfrac{1}{2}\{\cos(C-A+B)-\cos(C+A-B)\}.$$

On adding these three lines the terms on the right are seen to destroy each other.

EXAMPLES. XVI.

Express as a sum, (or difference), the following :

1. $2\sin(\alpha+\beta)\cos(\alpha-\beta)$.
2. $2\sin 3\alpha\cos\alpha$.
3. $2\sin 11^\circ\cos 27^\circ$.
4. $2\sin 2\theta\cos\theta$.
5. $2\cos 7A\cos 5A$.
6. $2\sin 9\theta\sin\theta$.

Prove the following results.

7. $\sin(\beta+\gamma)\sin(\beta-\gamma)+\sin(\gamma+\alpha)\sin(\gamma-\alpha)$
$+\sin(\alpha+\beta)\sin(\alpha-\beta)=0$.

8. $\sin(\beta-\gamma)\cos(\beta+\gamma)+\cos(\gamma-\alpha)\sin(\gamma+\alpha)$
$=\cos(\alpha-\beta)\sin(\alpha+\beta)$.

9. $\cos(120^\circ+A)\cos(120^\circ-A)=\dfrac{2\cos 2A-1}{4}$.

10. $\cos(30^\circ-A)\cos(60^\circ-A)=\dfrac{2\sin 2A+\sqrt{3}}{4}$.

11. $\sin(120^\circ-A)\cos(60^\circ+A)=\frac{1}{2}\sin(60^\circ-2A)$.

12. $\sin\left(\alpha+\dfrac{\pi}{4}\right)\sin\left(\alpha-\dfrac{\pi}{4}\right)=-\dfrac{\cos 2\alpha}{2}$.

13. $\sin\theta\sin 2\theta+\sin 3\theta\sin 6\theta=\sin 4\theta\sin 5\theta$.

14. $\sin 2A\cos A+\sin 6A\cos A=\sin 3A\cos 2A$
$+\sin 5A\cos 2A$.

15. $\sin 10^\circ.\cos 60^\circ+\sin 5^\circ\cos 45^\circ=\sin 15^\circ.\cos 55^\circ$.

16. $\cos(36^\circ-A)\cos(36^\circ+A)+\cos(54^\circ+A)\cos(54^\circ-A)$
$=\cos 2A$.

17. $\sin(n+1)B\sin(n-1)B+\cos(n+1)B\cos(n-1)B$
$=\cos 2B$.

18. $\sin A\cos(A+B)-\cos A\sin(A-B)=\cos 2A\sin B$.

19. $\cos\theta\sin(\phi-\psi)+\cos\phi\sin(\psi-\theta)+\cos\psi\sin(\theta-\phi)=0$.

20. $\cos(\beta+\gamma)\sin(\beta-\gamma)+\cos(\gamma+\alpha)\sin(\gamma-\alpha)$
$+\cos(\alpha+\beta)\sin(\alpha-\beta)=0.$

21. $\cos(A+B)\cos(A-B)-\cos(B+C)\cos(B-C)$
$+\cos(A+C)\cos(A-C)=\cos 2A.$

22. $\sin(\alpha+\beta)\cos\beta-\sin(\gamma+\alpha)\cos\gamma=\sin(\beta-\gamma)\cos(\alpha+\beta+\gamma).$

23. $1+\cos 2(A-B)\cos 2B=\cos^2 A+\cos^2(A-2B).$

24. $\sin(\alpha+\beta-2\gamma)\cos\beta-\sin(\alpha+\gamma-2\beta)\cos\gamma$
$=\sin(\beta-\gamma)\{\cos(\beta+\gamma-\alpha)+\cos(\gamma+\alpha-\beta)+\cos(\alpha+\beta-\gamma)\}.$

25. $\sin(A+B)\sin(B+C)=\sin A\sin C+\sin B\sin(A+B+C).$

26. Transform $4\sin A\sin B\sin C$ into a sum.

[$2\sin B\sin C=\cos(B-C)-\cos(B+C)$,

$\therefore 4\sin A\sin B\sin C=2\sin A\{\cos(B-C)-\cos(B+C)\}$
$=\sin(A+B-C)+\sin(A-B+C)-\sin(A+B+C)-\sin(A-B-C)$
$=\sin(A+B-C)+\sin(A-B+C)+\sin(B+C-A)-\sin(A+B+C)$.]

58. To prove that

$$\tan(\mathbf{A}+\mathbf{B})=\frac{\tan\mathbf{A}+\tan\mathbf{B}}{1-\tan\mathbf{A}\tan\mathbf{B}},$$

$$\tan(\mathbf{A}-\mathbf{B})=\frac{\tan\mathbf{A}-\tan\mathbf{B}}{1-\tan\mathbf{A}\tan\mathbf{B}}.$$

From the values already found for $\sin(A+B)$ and $\cos(A+B)$, the value of $\tan(A+B)$ may be found at once: thus

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B},$$

divide the numerator and denominator of the last expression by $\cos A\cos B$, we get

$$\tan(A+B)=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\cdot\dfrac{\sin B}{\cos B}}=\frac{\tan A+\tan B}{1-\tan A\tan B}\ldots\ldots\text{I.}$$

In like manner

$$\tan(A-B)=\frac{\sin(A-B)}{\cos A-B}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B},$$

hence, proceeding as before,

$$\tan(A-B)=\frac{\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}}{1+\frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}}=\frac{\tan A-\tan B}{1+\tan A\tan B}\text{......II.}$$

If in I. we take B equal to A we regain a formula already proved,

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}.$$

Ex. Find the value of $\tan(45°+A)$.

$$\tan(45°+A)=\frac{\tan 45°+\tan A}{1-\tan 45°\,.\,\tan A},\text{ but }\tan 45°=\text{unity},$$

hence, $$\tan(45°+A)=\frac{1+\tan A}{1-\tan A}.$$

EXAMPLES. . XVII.

1. If $\tan A=\frac{1}{4}$, and $\tan B=\frac{1}{5}$, show that

$$\tan(A+B)=\tfrac{9}{19}.$$

[Substitute for $\tan A$ and $\tan B$ in I. of this Article.]

2. Given that $\tan A=\frac{1}{2}$, $\tan B=\frac{1}{3}$, show that

$$\tan(A+B)=1.$$

3. If $\tan(A+B)=3$, and $\tan A=2$, what is $\tan B$?

4. Prove that $\alpha+\beta=\dfrac{\pi}{2}$, if $\tan\alpha=4$, and $\tan\beta=\dfrac{1}{4}$.

Prove the following results:

5. $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A+\cot B}$.

6. $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$.

7. $\tan(45^\circ+A)\tan(45^\circ-A)=1$.

8. $\tan(45^\circ+A)-\tan(45^\circ-A)=2\tan 2A$.

9. $\tan(A+30^\circ)=\dfrac{\sqrt{3}\tan A+1}{\sqrt{3}-\tan A}$.

10. $\tan(A+60^\circ)\tan(A-60^\circ)=\dfrac{1+2\cos 2A}{1-2\cos 2A}$.

11. $\tan(A+B)\tan(A-B)=\dfrac{\tan^2 A-\tan^2 B}{1-\tan^2 A\tan^2 B}$.

12. $\tan(A+B)+\tan(A-B)=\dfrac{\sin 2A\sec^2 A\sec^2 B}{1-\tan^2 A\tan^2 B}$.

13. $\tan(A+45^\circ)=\sqrt{\dfrac{1+\sin 2A}{1-\sin 2A}}$.

14. $1+\tan A\tan 2A=\tan 2A\cot A-1$.

15. $\sqrt{3}+\tan 40^\circ+\tan 80^\circ=\sqrt{3}\tan 40^\circ\,.\,\tan 80^\circ$.

16. If $\tan A=\dfrac{a}{b}$, $\tan B=\dfrac{c}{d}$,

$$\tan(A+B)=\frac{ad+bc}{bd-ac}.$$

17. If $\tan B=\dfrac{1-\tan A}{1+\tan A}$, then $\tan(A+B)=1$.

18. $\tan\alpha+\tan\beta+\tan(\pi-\alpha-\beta)$
$$=\tan\alpha\tan\beta\tan(\pi-\alpha-\beta).$$

59*. General proof of fundamental formulæ.

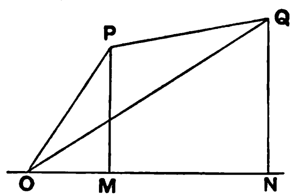
In finding the trigonometrical ratios of the sum and difference of A and B we have supposed that A, B, and $A+B$ are acute angles.

The method of proof adopted has the advantage of being, after a slight modification, applicable to angles of any size.

The modification consists in attaching a sign to a projection; thus in the figure of Article 51, PQ has the positive projection MN, but QP the negative projection NM, so that projections measured towards the right are positive, projections measured towards the left are negative.

With this understanding we easily see that the projection of PQ is $= PQ \cos A$, where A is the angle which PQ makes with the *positive direction* OA. So that if A is acute the projection is positive, if A is of the second quadrant, the projection is negative, and so on.

It follows that the projection of one side of a triangle is always equal to the sum of the projections of the other two sides *taken in order*, for example we notice in the figure



that the projection of OP = projection of OQ + projection of QP.

60. We proceed to find $\cos(A+B)$, where both A and B are obtuse.

Call the angles AOC, COD respectively A and B, and project on OA. Draw PK perpendicular to CO produced.

Then projection OP = projection OK + projection KP

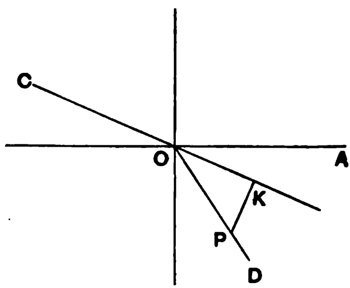
and the angle which $\begin{cases} OP \text{ makes with } OA \text{ is } A+B, \\ OK \text{ makes with } OA \text{ is } 180^\circ + A, \\ KP \text{ makes with } OA \text{ is } 90^\circ + A; \end{cases}$

also $$OK = OP \cos(180° - B) = - OP \cos B$$
$$KP = OP \sin(180° - B) = OP \sin B.$$

Hence

$$OP \cos(A + B) = - OP \cos B \cos(180° + A) + OP \sin B \cos(90° + A),$$

or $$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$



In like manner the other formulæ in Articles 53 and 54 may be proved for obtuse angles. Those results are therefore proved generally, and thus also the results of Article 56, which depend on them.

61. Other proofs of the formulæ for sin (A ± B) and cos (A ± B).

The following method of proof, which used commonly to be given in works on trigonometry, may be regarded as an alternative to that which we have adopted.

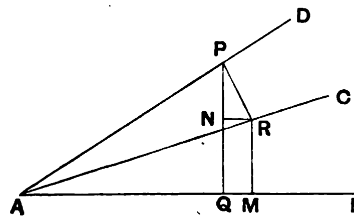
In the figure on the next page, let the angle $BAC = A$, and the angle $CAD = B$, then the angle $BAD = A + B$.

From P, any point on the line bounding the angle whose trigonometrical ratios are required, draw PQ and PR perpendicular to AB and AC.

From R draw RM and RN perpendicular to AB and PQ.

The angle $NPR = 90° - \angle PRN = \angle NRA = A$;

hence $$\frac{NP}{PR} = \cos A, \quad \frac{NR}{PR} = \sin A.$$



We have

$$\sin(A+B) = \sin BAD = \frac{PQ}{AP} = \frac{QN+NP}{AP} = \frac{RM}{AP} + \frac{NP}{AP},$$

$$\therefore \sin(A+B) = \frac{RM}{AR} \cdot \frac{AR}{AP} + \frac{NP}{PR} \cdot \frac{PR}{AP}$$

$$= \sin A \cos B + \cos A \sin B.$$

Again

$$\cos(A+B) = \cos BAD = \frac{AQ}{AP} = \frac{AM-QM}{AP} = \frac{AM-NR}{AP},$$

$$\therefore \cos(A+B) = \frac{AM}{AR} \cdot \frac{AR}{AP} - \frac{NR}{PR} \cdot \frac{PR}{AP}$$

$$= \cos A \cos B - \sin A \sin B.$$

62. We find the sine and cosine of $A - B$ in a similar manner.

Let the angle $BAC = A$, and the angle $CAD = B$, then will angle $BAD = A - B$.

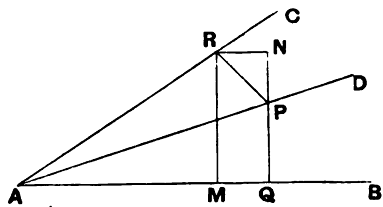
From P any point on the line bounding the angle whose

trigonometrical ratios are required, draw PQ and PR perpendicular to AB and AC. Through R draw RM and RN perpendicular to AB and PQ.

The angle

$$NPR = 90° - NRP = NRC = A;$$

hence $$\frac{PN}{PR} = \cos A, \quad \frac{NR}{PR} = \sin A.$$



$$\sin(A-B) = \sin BAD = \frac{PQ}{AP} = \frac{QN - PN}{AP} = \frac{RM}{AP} - \frac{PN}{AP}.$$

$$\therefore \sin(A-B) = \frac{RM}{AR} \cdot \frac{AR}{AP} - \frac{PN}{PR} \cdot \frac{PR}{AP}$$

$$= \sin A \cdot \cos B - \cos A \sin B.$$

Again

$$\cos(A-B) = \cos BAD = \frac{AQ}{AP} = \frac{AM + MQ}{AP} = \frac{AM}{AP} + \frac{NR}{AP}.$$

$$\therefore \cos(A-B) = \frac{AM}{AR} \cdot \frac{AR}{AP} + \frac{NR}{PR} \cdot \frac{PR}{AP}$$

$$= \cos A \cos B + \sin A \cdot \sin B.$$

63. Various Formulae.

We add for reference a number of miscellaneous

formulae which are frequently useful, most of them have already been proved.

$$\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B.$$

$$\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A.$$

$$\tan(45^\circ+A)=\frac{1+\tan A}{1-\tan A},$$

$$\tan(45^\circ-A)=\frac{1-\tan A}{1+\tan A}.$$

$$\sin 3A = 3\sin A-4\sin^3 A.$$

$$\cos 3A = -3\cos A+4\cos^3 A.$$

$$\tan 3A = \frac{3\tan A-\tan^3 A}{1-3\tan^2 A}.$$

$$\sin 15^\circ=\frac{\sqrt{6}-\sqrt{2}}{4},\quad \cos 15^\circ=\frac{\sqrt{6}+\sqrt{2}}{4},$$

$$\tan 15^\circ=2-\sqrt{3},\qquad \cot 15^\circ=2+\sqrt{3}.$$

$$\sin 18^\circ=\frac{\sqrt{5}-1}{4},\qquad \cos 18^\circ=\frac{\sqrt{10+2\sqrt{5}}}{4},$$

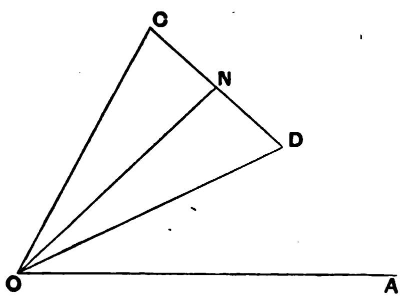
$$\tan 18^\circ=\tfrac{1}{5}\sqrt{25-10\sqrt{5}},\quad \cot 18^\circ=\sqrt{5+2\sqrt{5}}.$$

CHAPTER VIII.

SPECIAL GEOMETRICAL PROOFS.

64. The product formulæ.

Take two equal lines OC and OD such that $\angle AOC = C$, $\angle AOD = D$. Join CD and draw ON perpendicular to CD, N will be the middle point of CD. Also $\angle NOA = \frac{1}{2}(C + D)$, $\angle NOC = \frac{1}{2}(C - D) = \angle NOD$.



Project on OA.

Projection of OC = projection of ON − projection of CN.

Projection of OD = projection of ON + projection of ND;

hence, projection OC + projection of OD

= twice projection of ON;

since projection of CN = projection of ND.

Or, $$OC\cos C + OD\cos D = 2ON\cos\frac{C+D}{2},$$

but $$ON = OC\cos\frac{C-D}{2}, \text{ therefore}$$

$$\cos C + \cos D = 2\cos\frac{C-D}{2}\cos\frac{C+D}{2}.$$

Next projecting on OB, the perpendicular to OA, we get in like manner,

$$OC\sin C + OD\sin D = 2ON\sin\frac{C+D}{2},$$

or, $$\sin C + \sin D = 2\cos\frac{C-D}{2}\sin\frac{C+D}{2}.$$

Again, projecting on OA,

projection OD = projection of OC + twice projection of CN, since $CN = ND$,

also $$CN = OC\sin\frac{C-D}{2}, \text{ hence}$$

$$OD\cos D = OC\cos C + 2OC\sin\frac{C-D}{2}\sin\frac{C+D}{2},$$

or, $$\cos D = \cos C + 2\sin\frac{C-D}{2}\sin\frac{C+D}{2}.$$

Similarly projecting on OB,

$$OC\sin C = OD\sin D + 2OC\sin\frac{C-D}{2}\cos\frac{C+D}{2},$$

or, $$\sin C = \sin D + 2\sin\frac{C-D}{2}\cos\frac{C+D}{2}.$$

65. To prove geometrically

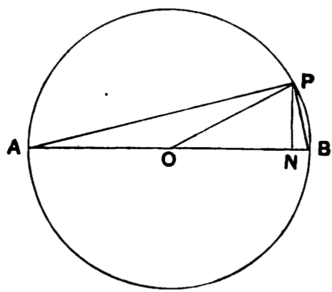
$$\sin 2A = 2\sin A\cos A, \quad \cos 2A = \cos^2 A - \sin^2 A.$$

Let AOB be the diameter of a circle, and let $PAB = A$, then $POB = 2A$; draw PN perpendicular to AB.

Then $$\sin 2A = \frac{PN}{OP},$$

now $PN \,.\, AB = 2 \triangle APB = AP \,.\, PB$, since $\angle APB$ is 90°; therefore

$$\sin 2A = \frac{PN \,.\, AB}{OP \,.\, AB} = \frac{AP \,.\, PB}{OP \,.\, AB} = \frac{AB^2 \cos A \sin A}{OP \,.\, AB}$$
$$= 2 \sin A \cos A, \text{ since } OP = \tfrac{1}{2}AB;$$



also $$\cos 2A = \frac{ON}{OP} = \frac{ON \,.\, AB}{OP \,.\, AB} = \frac{AN^2 - BN^2}{2OP \,.\, AB} = \frac{AP^2 - BP^2}{AB^2}$$
$$= \cos^2 A - \sin^2 A.$$

Notice that,

$$\tan A = \frac{PN}{AN} = \frac{PN}{AO + ON} = \frac{\frac{PN}{OP}}{1 + \frac{ON}{OP}} = \frac{\sin 2A}{1 + \cos 2A}.$$

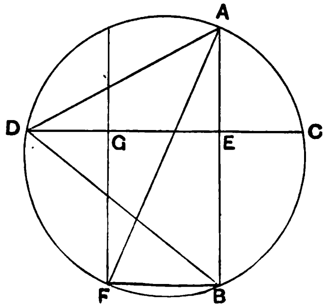
66. To prove geometrically that

$$\tan (\mathbf{A} + \mathbf{B}) = \frac{\tan \mathbf{A} + \tan \mathbf{B}}{1 - \tan \mathbf{A} \tan \mathbf{B}}.$$

AB and *CD* are two perpendicular chords of a circle; draw *BF* parallel to *CD*, and through *F* draw *FG* parallel to *AB*. Since $CE = GD$, we see that $ED - EC = EG = BF$.

Let the angles ADE, BDE be denoted by A and B; then $\angle AFB = A + B$, since it is on the same arc AB, as ADB.

Hence $\tan(A+B) = \dfrac{AB}{BF} = \dfrac{AE+EB}{ED-EC} = \dfrac{(AE+EB)\,ED}{(ED-EC).ED}$,

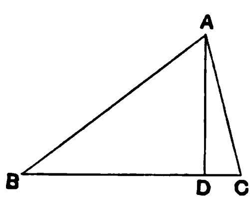


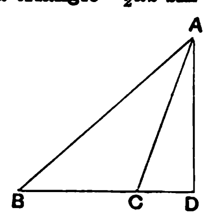
but $\qquad EC \,.\, ED = EA \,.\, EB$, hence

$$\tan(A+B) = \frac{(AE+EB)\,ED}{ED^2 - EA\,.\,EB} = \frac{\dfrac{AE}{ED} + \dfrac{EB}{ED}}{1 - \dfrac{EA}{ED}\cdot\dfrac{EB}{ED}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A\,.\,\tan B}.$$

67. To prove that the area of a triangle $= \frac{1}{2}ab \sin C$.





Let ABC be a triangle, then as has been already explained, it is usual to denote the angles at the vertices A, B

and C by those letters, the lengths of the sides BC, CA and AB being denoted by a, b and c respectively.

From A draw AD perpendicular to BC.

$$\text{Area of } \triangle\, ABC = \tfrac{1}{2} AD \,.\, BC,$$

$$\text{but } AD = b \sin C, \text{ and } BC = a, \text{ hence}$$

$$\text{area of } \triangle\, ABC = \tfrac{1}{2} ab \sin C.$$

Similarly we might show its area to be $\frac{1}{2}bc \sin A$, or $\frac{1}{2}ac \sin B$.

68. To prove geometrically

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Using the figure of the last Article, let us call $\angle\, BAD$, α and the $\angle\, CAD$, β. Then $AD = b\cos\beta = c\cos\alpha$.

Area of $\triangle\, ABC$ = area of $\triangle\, BAD$ + area of $\triangle\, CAD$,

$$\text{and} \begin{cases} \text{Area of triangle } ABC = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}bc\sin(\alpha+\beta). \\ \text{Area of triangle } BAD = \tfrac{1}{2}AD \,.\, BD = \tfrac{1}{2}b\cos\beta \,.\, c\sin\alpha. \\ \text{Area of triangle } CAD = \tfrac{1}{2}AD \,.\, CD = \tfrac{1}{2}c\cos\alpha \,.\, b\sin\beta; \end{cases}$$

therefore $\frac{1}{2}bc\sin(\alpha+\beta) = \frac{1}{2}bc\cos\beta\sin\alpha + \frac{1}{2}bc\cos\alpha\sin\beta$,

from which the result follows. This theorem has been already proved geometrically in Article 54.

MISCELLANEOUS EXAMPLES. XVIII.

Prove the following identities

1. $\cos A + \cos(120^\circ - A) + \cos(120^\circ + A) = 0.$

2. $\cos^3 A - \sin^3 A = \sqrt{2}\cos(45^\circ + A)(1 + \sin A\cos A).$

3. $\sin 3A = \sin A\cos 2A + \cos A \sin 2A$
$$= 3\sin A - 4\sin^3 A.$$

4. $\cos 3A = \cos A\cos 2A - \sin A\sin 2A$
$$= -3\cos A + 4\cos^3 A.$$

5. $$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{A+B}{2}}{\tan \frac{A-B}{2}}.$$

6. $$\frac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{A+B}{2}.$$

7. $$\frac{\sin A - \sin B}{\cos A + \cos B} = \tan \frac{A-B}{2}.$$

8. $$\frac{\cos A + \cos B}{\cos B - \cos A} = \cot \frac{A+B}{2} \cot \frac{A-B}{2}.$$

9. $(\sin 2A - \sin 2B) \tan (A+B) = 2 (\sin^2 A - \sin^2 B)$.

10. $$\frac{\sin 2A}{1 + \sin 2A} = \frac{2}{(1 + \tan A)(1 + \cot A)}.$$

11. $2 \cos (A + 30^\circ) \cos (A + 120^\circ) = \cos (2A + 150^\circ)$.

12. $\cos^2 A - \cos^2 3A = \sin 4A \sin 2A$.

13. $1 + \cos^2 2A = 2 (\sin^4 A + \cos^4 A)$.

14. $$\sin \theta + \sin \left(\theta + \frac{2\pi}{3}\right) + \sin \left(\theta + \frac{4\pi}{3}\right) = 0.$$

15. $$\frac{\sin (x + 3y) + \sin (3x + y)}{\sin 2x + \sin 2y} = 2 \cos (x + y).$$

16. $$\tan \left(A + \frac{\pi}{4}\right) = \frac{\cos 2A}{1 - \sin 2A}.$$

17. $$\cos (B + C) \cos (B - C) - \cos (A + C) \cos (A - C) = \sin (A + B) \sin (A - B)$$

18. If $\tan A = \dfrac{\sqrt{3}}{4 - \sqrt{3}}$, $\tan B = \dfrac{\sqrt{3}}{4 + \sqrt{3}}$, then

$$\tan (A - B) = \cdot 375.$$

19. $\sin 70^\circ = \sin 10^\circ + \sin 50^\circ$.

20. $\sin 118^\circ - \sin 2^\circ = \sin 58^\circ$.

21. $\cos 9° - \cos 51° = \sin 21°.$

22. $\sin(A-B)\sin(C+D)+\sin(B-C)\sin(A+D)$
$+\sin(C-A)\sin(B+D)=0.$

23. $\dfrac{1+\sin A}{1+\cos A}=\frac{1}{2}\left(1+\tan\dfrac{A}{2}\right)^2.$

24. $\cot\dfrac{\pi}{8}-\tan\dfrac{\pi}{8}=2.$

25. $\sin(A+B)-\dfrac{\sin(2A+B)-\sin B}{2\cos A}=\dfrac{\sin B}{\cos A}.$

26. $\dfrac{\sin(\beta+\gamma)-\sin\beta}{\sin(\alpha+\gamma)-\sin\alpha}=\dfrac{2\cos\dfrac{\beta+\gamma}{2}\cos\dfrac{\beta}{2}-\cos\dfrac{\gamma}{2}}{2\cos\dfrac{\alpha+\gamma}{2}\cos\dfrac{\alpha}{2}-\cos\dfrac{\gamma}{2}}.$

27. $\sin(A+C)\cos(A-C)-\sin(B+C)\cos(B-C)$
$=\cos(A+B)\sin(A-B).$

28. $\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta\cos(\alpha+\beta)=\sin^2(\alpha+\beta).$

29. $\operatorname{cosec}2\theta+\cot 4\theta+\operatorname{cosec}4\theta=\cot\theta.$

30. $\cot\theta-\tan\theta=2\sqrt{\dfrac{1+\cos 4\theta}{1-\cos 4\theta}}.$

31. $\{\sin A+\sin B+\sin(A+B)\}^2$
$+\{1+\cos A+\cos B+\cos(A+B)\}^2=16\cos^2\dfrac{A}{2}\cos^2\dfrac{B}{2}.$

32. $\tan^4\theta=\dfrac{2\tan\theta-\sin 2\theta}{2\cot\theta-\sin 2\theta}.$

33. $\tan(A+60°)\tan(A-60°)+\tan A\tan(A+60°)$
$+\tan(A-60°)\tan A=-3.$

34. $\cot(A+60°)\cot(A-60°)+\cot A\cot(A+60°)$
$+\cot(A-60°)\cot A=-3.$

35. $\cos(A+B)\sin B-\cos(A+C)\sin C$
$=\sin(A+B)\cos B-\sin(A+C)\cos C.$

36. $1 + \cos 2(A - B)\cos 2B = \cos^2 A + \cos^2 (A - 2B)$.

37. $$\sin\theta = \frac{\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) - \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}.$$

38. $\cos^4 B - \cos^4 A = \sin(A + B)\sin(A - B)$
$\{1 + \cos(A + B)\cos(A - B)\}$.

39. $$\left\{\cot\theta + \cot\left(\theta - \frac{\pi}{2}\right)\right\}\left\{\tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{\pi}{4} + \theta\right)\right\} = \frac{4}{\sin 2\theta}.$$

40. $\cos^6 A + \sin^6 A = 1 - \frac{3}{4}\sin^2 2A$.

41. If A, B and C are in A. P.,
$\sin A - \sin C = 2\sin(A - B)\cos B$.

42. $$\frac{\sin(A + 30^\circ) + \sin(B - 30^\circ)}{\cos A - \cos B} = \frac{\sqrt{3}}{2}\cot\frac{B - A}{2} + \frac{1}{2}.$$

43. $$\frac{\sin 7\theta - \sin 3\theta}{\cos 3\theta - \cos 7\theta} = \cot 5\theta.$$

44. $\sin(\alpha + \beta)\sin\beta + \cos(\gamma - \beta)\cos(\alpha + \beta + \gamma)$
$= \cos(\alpha + \gamma)\cos\gamma$.

45. $\dfrac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \tan\dfrac{\theta}{2}$. Hence find $\tan 15^\circ$ and $\tan 22\frac{1}{2}^\circ$.

46. $4\cos(A + B + 45^\circ)\cos(A + B - 45^\circ)\cos(A - B)$
$= \cos(A + 3B) + \cos(3A + B)$.

47. $\cos^2 A + \cos^2(120^\circ + A) + \cos^2(120^\circ - A) = \frac{3}{2}$.

48. $(\cos A + \sin A)^4 + (\cos A - \sin A)^4 = 3 - \cos 4A$.

49. $\sin A + \sin 2A + \sin 3A = \sin 2A(1 + 2\cos A)$.

50. $\sin A + 2\sin 3A + \sin 5A = 4\cos^2 A \sin 3A$.

51. $4\sin 20^\circ . \sin 40^\circ . \sin 80^\circ = \sin 60^\circ$. [See Ex. 26, page 91.]

52. $\dfrac{\sin A + \sin 2A + \sin 3A}{\cos A + \cos 2A + \cos 3A} = \tan 2A.$

53. $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = \tan 3\theta.$

54. $\cos 47^\circ - \cos 61^\circ - \cos 11^\circ + \cos 25^\circ = \sin 7^\circ.$

55. $1 + \tan 65^\circ + \tan 70^\circ = \tan 65^\circ . \tan 70^\circ.$

56. $4 \sin 110^\circ . \sin 70^\circ . \sin 40^\circ = \sin 80^\circ + 2 \sin 40^\circ.$

57. $\cos 40^\circ + \cos 80^\circ + \cos 160^\circ = 0.$

$$\cos 40^\circ . \cos 80^\circ + \cos 80^\circ . \cos 160^\circ + \cos 160^\circ . \cos 40^\circ = -\tfrac{3}{4}.$$

$\cos 40^\circ . \cos 80^\circ . \cos 160^\circ = -\tfrac{1}{8}.$

58. If the cosines of the angles

$$\beta + \frac{\gamma - \alpha}{2}, \quad \frac{\gamma + \alpha}{2}, \quad \beta - \frac{\gamma - \alpha}{2}$$

are in G. P., so are also the sines of

$$\tfrac{1}{2}(\alpha + \gamma) - \beta, \quad \tfrac{1}{2}(\gamma - \alpha), \quad \tfrac{1}{2}(\alpha + \gamma) + \beta.$$

59. $\cos(A - 2B) - \cos(A - B) + \cos A - \cos(A + B)$

$$+ \cos(A + 2B) = \cos A \frac{\cos \frac{5}{2}B}{\cos \frac{1}{2}B}$$

$$= 4 \cos A (\cos B - \cos 36^\circ)(\cos B + \cos 72^\circ).$$

60. $\dfrac{\sin 8A}{2 \sin A} = \cos A + \cos 3A + \cos 5A + \cos 7A.$

61. $\sin A (\cos 2A + \cos 4A + \cos 6A) = \sin 3A \cos 4A.$

62. $\sin A + \sin 2A + \sin 3A + \sin 4A = 4 \cos \dfrac{A}{2} \cos A \sin \dfrac{5A}{2}.$

63. $\sin A + \sin 3A + \sin 5A + \sin 7A$

$$= 4 \cos A \cos 2A \sin 4A.$$

CHAPTER IX*.

THE TRIGONOMETRICAL RATIOS OF THE SUM OF THREE ANGLES.

69. IF in the formula

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

we write $B+C$ instead of B, we have

$$\begin{aligned}\sin(A+B+C) &= \sin A \cos(B+C) + \cos A \sin(B+C)\\ &= \sin A(\cos B \cos C - \sin B \sin C)\\ &\qquad + \cos A(\sin B \cos C + \cos B \sin C),\end{aligned}$$

hence we have the formula

$$\begin{aligned}\sin(A+B+C) = {} & \sin A \cos B \cos C + \sin B \cos C \cos A\\ & + \sin C \cos A \cos B - \sin A \sin B \sin C \;\ldots\ldots\ldots(1).\end{aligned}$$

In a similar manner we have

$$\begin{aligned}\cos(A+B+C) &= \cos A \cos(B+C) - \sin A \sin(B+C)\\ &= \cos A(\cos B \cos C - \sin B \sin C)\\ &\qquad - \sin A(\sin B \cos C + \cos B \sin C).\end{aligned}$$

Thus

$$\begin{aligned}\cos(A+B+C) = {} & \cos A \cos B \cos C - \cos A \sin B \sin C\\ & - \cos B \sin C \sin A - \cos C \sin A \sin B \;\ldots\ldots\ldots(2).\end{aligned}$$

The formulae (1) and (2) give the sine and the cosine of the sum of three angles in terms of the sines and cosines of the angles.

The formulae (1) and (2) may also be written

$$\sin(A+B+C)$$
$$=\cos A\cos B\cos C\,(\tan A+\tan B+\tan C-\tan A\tan B\tan C),$$
$$\cos(A+B+C)$$
$$=\cos A\cos B\cos C\,(1-\tan B\tan C-\tan C\tan A-\tan A\tan B),$$

hence since

$$\tan(A+B+C)=\frac{\sin(A+B+C)}{\cos(A+B+C)},$$

we have the formula

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B} \quad \ldots\ldots\ldots\ldots(3).$$

This formula expresses the tangent of the sum of three angles in terms of the tangents of those angles.

Observe that if $A+B+C=180°$,

$$\sin(A+B+C)=0,$$

and therefore

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

This result will be proved directly in the sequel.

Also if

$$A+B+C=90°, \quad \text{we have } \cos(A+B+C)=0,$$

and then

$$1=\tan B\tan C+\tan C\tan A+\tan A\tan B.$$

70. To prove the formulae

$$\sin 3A=3\sin A-4\sin^3 A,$$
$$\cos 3A=-3\cos A+4\cos^3 A.$$

These results may be obtained by putting B and C each

equal to A in (1) and (2). They are however most easily proved independently as follows:

$$\sin 3A = \sin A \cos 2A + \cos A \sin 2A$$
$$= \sin A\,(1 - 2\sin^2 A) + \cos A\,.\,2\sin A\cos A$$
$$= \sin A - 2\sin^3 A + 2\sin A\,(1 - \sin^2 A),$$

or
$$\sin 3A = 3\sin A - 4\sin^3 A \text{......................}(4).$$

Also
$$\cos 3A = \cos A\cos 2A - \sin A\sin 2A$$
$$= \cos A\,(2\cos^2 A - 1) - \sin A\,.\,2\sin A\cos A$$
$$= 2\cos^3 A - \cos A - 2\cos A\,(1 - \cos^2 A)$$

or
$$\cos 3A = 4\cos^3 A - 3\cos A \text{ .... }(5).$$

We get from (4) and (5) by division,

$$\tan 3A = \frac{3\sin A - 4\sin^3 A}{4\cos^3 A - 3\cos A} = \frac{3\sin A\,(\sin^2 A + \cos^2 A) - 4\sin^3 A}{4\cos^3 A - 3\cos A\,(\sin^2 A + \cos^2 A)},$$

hence
$$\tan 3A = \frac{3\sin A\cos^2 A - \sin^3 A}{\cos^3 A - 3\cos A\sin^2 A},$$

divide numerator and denominator by $\cos^3 A$, we get

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \text{ ....}(6).$$

EXAMPLES. XIX.

1. Prove that $\sin 3A = \sin A\,(2\cos 2A + 1)$.

2. Prove that $\cos 3A = \cos A\,(2\cos 2A - 1)$.

3. From these formulae deduce the value of $\sin 15°$ and $\cos 15°$.

4. Find $\cos 3A$ when $\cos A = \frac{1}{3}$.

5. Find $\sin 3A$ when $\sin A = \frac{1}{4}$.

Prove the following results:

6. $\sin 3A = 4\sin A\sin(60° + A)\sin(60° - A)$.

7. $\cos 3A = 4 \cos A \cos (60° + A) \cos (60° - A)$.

8. $\dfrac{\sin 3A}{\sin A} - \dfrac{\sin 3B}{\sin B} = 4 \sin (A + B) \sin (B - A)$.

9. $\dfrac{\sin 3A + \sin A}{\cos A - \cos 3A} = \dfrac{\sin 3A + \sin^3 A}{\cos^3 A - \cos 3A} = \cot A$.

10. $\cot 3A = \dfrac{\cot^3 A - 3 \cot A}{3 \cot^2 A - 1}$.

11. $\sin^3 A + \sin^3 (120° + A) - \sin^3 (120° - A) = -\frac{3}{4} \sin 3A$.

[Substitute for $\sin^3 A$ its value in terms of $\sin 3A$ and $\sin A$.]

12. $\sin 3A \sin^3 A + \cos 3A \cos^3 A = \cos^3 2A$.

13. $4 \cos^3 A \sin 3A + 4 \sin^3 A \cos 3A = 3 \sin 4A$.

14. $\sin 3A \cos^3 A + \cos 3A \sin^3 A$

$$= \frac{1}{\sqrt{2}} \sin 2A \sin \left(\frac{\pi}{4} - A\right) \{3 + 2 \sin 2A\}.$$

15. $\dfrac{\frac{1}{2}}{\sin 10°} - \dfrac{\frac{\sqrt{3}}{2}}{\cos 10°} = 2$.

16. $4 (\cos^3 10° + \sin^3 20°) = 3 (\cos 10° + \sin 20°)$.

17. $5 + 3 \cos 4A = 8 (\cos^6 A + \sin^6 A)$.

18. $\cos^3 2\theta + 3 \cos 2\theta = 4 (\cos^6 \theta - \sin^6 \theta)$.

19. $\cos 3 \left(A + \dfrac{\pi}{4}\right) + \cos \left(A + \dfrac{\pi}{4}\right) \{1 + 2 \sin 2A\} = 0$.

20. $\tan A + \tan 2A - \tan 3A + \tan A \tan 2A \tan 3A = 0$.

21. $\tan (B - C) + \tan (C - A) + \tan (A - B)$

$$= \tan (B - C) \tan (C - A) \tan (A - B).$$

22. $\tan mA + \tan nA + \tan rA = \tan mA \tan nA \tan rA$

if $$m + n + r = 0.$$

71. To find sin 18°.

The angles 36° and 54° are complementary, and therefore if we write A for an angle of 18°,

$$\sin 2A = \cos 3A \qquad \text{Article 19};$$

hence $$2 \sin A \cos A = -3 \cos A + 4 \cos^3 A,$$

and omitting the factor $\cos A$,

$$2 \sin A = -3 + 4 \cos^2 A,$$

putting $1 - \sin^2 A$ for $\cos^2 A$, this equation may be written

$$4 \sin^2 A + 2 \sin A - 1 = 0.$$

This is a quadratic equation to find $\sin A$.

Solving it we have

$$\sin A = \frac{-1 \pm \sqrt{5}}{4}.$$

Now sin 18° is a positive quantity, we therefore take the upper sign, hence

$$\sin 18° = \frac{\sqrt{5} - 1}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

And

$$\cos 18° = \sqrt{1 - \sin^2 18°} = \sqrt{1 - \frac{(\sqrt{5} - 1)^2}{16}},$$

or, $$\cos 18° = \frac{\sqrt{10 + 2\sqrt{5}}}{4} \quad \ldots\ldots\ldots\ldots\ldots (8).$$

72. To find sin 3°.

We have seen already that

$$\sin 15° = \sin (45° - 30°) = \frac{\sqrt{3} - 1}{2\sqrt{2}},$$

$$\cos 15° = \cos (45° - 30°) = \frac{\sqrt{3} + 1}{2\sqrt{2}}.$$

Hence

$$\sin 3° = \sin (18° - 15°) = \sin 18° \cos 15° - \cos 18° \sin 15°,$$

or $$\sin 3° = \frac{(\sqrt{5} - 1)(\sqrt{3} + 1) - \sqrt{10 + 2\sqrt{5}}\,(\sqrt{3} - 1)}{8\sqrt{2}} \quad \ldots (9),$$

similarly

$$\cos 3^\circ = \frac{\sqrt{10+2\sqrt{5}}\,(\sqrt{3}+1)+(\sqrt{5}-1)\,(\sqrt{3}-1)}{8\sqrt{2}} \quad \ldots(10).$$

Now we have

$$6^\circ = 36^\circ - 30^\circ, \qquad 9^\circ = 45^\circ - 36^\circ, \qquad 12^\circ = 30^\circ - 18^\circ,$$

$$21^\circ = 36^\circ - 15^\circ, \qquad 24^\circ = 45^\circ - 21^\circ, \qquad 27^\circ = 30^\circ - 3^\circ,$$

$$33^\circ = 45^\circ - 12^\circ, \qquad 39^\circ = 45^\circ - 6^\circ, \qquad 42^\circ = 45^\circ - 3^\circ,$$

hence we can calculate the sines and cosines of the angles

$$3^\circ,\ 6^\circ,\ 9^\circ \ldots\ldots \text{ up to } 45^\circ;$$

we need not proceed further since the sine or cosine of an angle greater than 45° is the cosine or sine of its complement, which is less than 45°.

Ex. Prove that $\sin\frac{\pi}{5} \cdot \sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$.

73. To express as a product;

(I) $\mathbf{\sin A + \sin B + \sin C - \sin(A+B+C)}$.

(II) $\mathbf{\cos^2 A + \cos^2 B + \cos^2 C - 2\cos A\cos B\cos C - 1}$.

I. By Article 56,

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2},$$

$$\sin(A+B+C) - \sin C = 2\cos\frac{A+B+2C}{2}\sin\frac{A+B}{2},$$

hence subtracting

$$\sin A + \sin B + \sin C - \sin(A+B+C)$$

$$= 2\sin\frac{A+B}{2}\left\{\cos\frac{A-B}{2} - \cos\frac{A+B+2C}{2}\right\}$$

$$= 4\sin\frac{A+B}{2}\sin\frac{A+C}{2}\sin\frac{B+C}{2}$$

$$= 4\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2} \quad \ldots\ldots\ldots\ldots\ldots(11).$$

II. We have

$$\cos^2 A + \cos^2 B - 1 = \cos(A+B)\cos(A-B)$$

Article 63,

and $\quad 2\cos A\cos B\cos C = \cos C\{\cos(A+B) + \cos(A-B)\}$,

hence

$$\begin{aligned}&\cos^2 A + \cos^2 B - 1 - 2\cos A\cos B\cos C + \cos^2 C\\ &\quad = \cos(A+B)\cos(A-B) - \cos C\{\cos(A+B)\\ &\qquad\qquad + \cos(A-B)\} + \cos^2 C\\ &\quad = \{-\cos(A+B) + \cos C\}\{-\cos(A-B) + \cos C\}.\end{aligned}$$

Each of the quantities in brackets can be put in factors, thus

$$\cos C - \cos(A+B) = 2\sin\frac{A+B+C}{2}\sin\frac{A+B-C}{2},$$

$$\cos C - \cos(A-B) = 2\sin\frac{A-B+C}{2}\sin\frac{A-B-C}{2},$$

from which it follows that

$$\begin{aligned}&\cos^2 A + \cos^2 B + \cos^2 C - 2\cos A\cos B\cos C - 1\\ &= -4\sin\frac{A+B+C}{2}\sin\frac{A+B-C}{2}\sin\frac{A-B+C}{2}\sin\frac{-A+B+C}{2}\end{aligned}$$

.........(12).

Observe that if $A + B + C = 180°$ we have from I. that

$$\sin A + \sin B + \sin C = 4\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}$$

$$= 4\cos\frac{C}{2}\cos\frac{B}{2}\cos\frac{A}{2}\ \ldots\ldots\ldots\ldots(13),$$

since $\quad \sin\dfrac{A+B}{2} = \sin\left(90° - \dfrac{C}{2}\right) = \cos\dfrac{C}{2}$, &c.

From II. we have that if $A + B + C = 360°$,

$$\cos^2 A + \cos^2 B + \cos^2 C - 2\cos A\cos B\cos C - 1 = 0\ \ldots(14).$$

74. Relations which hold when A+B+C=180°.

By using the rules of Article 56, we can often factorize expressions involving trigonometrical ratios of A, B and C, in the case where $A+B+C=180°$. A few such cases are added. In all of them we suppose $A+B+C=180°$.

(i) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2},$$

$$\sin C=\sin(180°-A-B)=\sin(A+B)=2\sin\frac{A+B}{2}\cos\frac{A+B}{2},$$

$$\therefore\ \sin A+\sin B+\sin C=2\sin\frac{A+B}{2}\left\{\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right\}$$

$$=2\cos\frac{C}{2}\,.\,2\cos\frac{A}{2}\cos\frac{B}{2}$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

(ii) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$.

By the method of last example

$$\sin 2A+\sin 2B=2\sin(A+B)\cos(A-B),$$

$$\sin 2C=\sin(360°-2A-2B)=-\sin(2A+2B)$$
$$=-2\sin(A+B)\cos(A+B),$$

$$\therefore\ \sin 2A+\sin 2B+\sin 2C$$
$$=2\sin(A+B)\{\cos(A-B)-\cos(A+B)\}$$
$$=2\sin C\,.\,2\sin A\sin B$$
$$=4\sin A\sin B\sin C.$$

(iii) $\cos A+\cos B+\cos C-1=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

We have

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} = 2\sin\frac{C}{2}\cos\frac{A-B}{2},$$

$$\cos C - 1 = -2\sin^2\frac{C}{2},$$

$$\therefore \cos A + \cos B + \cos C - 1 = 2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2} - \sin\frac{C}{2}\right\}$$

$$= 2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right\}$$

$$= 4\sin\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2}.$$

(iv) $\cos A - \cos B + \cos C + 1 = 4\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$.

$$\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2} = 2\cos\frac{C}{2}\sin\frac{B-A}{2},$$

$$\cos C + 1 = 2\cos^2\frac{C}{2},$$

$$\therefore \cos A - \cos B + \cos C + 1 = 2\cos\frac{C}{2}\left\{\sin\frac{B-A}{2} + \cos\frac{C}{2}\right\}$$

$$= 2\cos\frac{C}{2}\left\{\sin\frac{B-A}{2} + \sin\frac{B+A}{2}\right\}$$

$$= 4\cos\frac{C}{2}\sin\frac{B}{2}\cos\frac{A}{2}.$$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

By Article 58, since

$$\tan(A+B) = \tan(180^\circ - C) = -\tan C,$$

$$\tan C = -\frac{\tan A + \tan B}{1 - \tan A \tan B}$$

or $$\tan C(1 - \tan A \tan B) + \tan A + \tan B = 0.$$

EXAMPLES. XX.

Prove the following, where $A + B + C = 180°$.

1. $\sin 4A + \sin 4B + \sin 4C = -4 \sin 2A \sin 2B \sin 2C$.

2. $\sin 3A + \sin 3B + \sin 3C = -4 \cos \frac{3A}{2} \cos \frac{3B}{2} \cos \frac{3C}{2}$.

3. $\cos 2A + \cos 2B + \cos 2C + 1 = -4 \cos A \cos B \cos C$.

4. $\cos 2A + \cos 2B + \sin 2C$
$$= 4 \cos C \cos \left(\frac{\pi}{4} + A\right) \sin \left(B - \frac{\pi}{4}\right).$$

5. $\cos 2A - \cos 2B + \sin 2C$
$$= -4 \sin C \sin \left(\frac{\pi}{4} + A\right) \cos \left(\frac{\pi}{4} + B\right).$$

6. $\sin A + \sin B + \cos C + 1$
$$= 4 \cos \frac{C}{2} \cos \left(\frac{\pi}{4} - \frac{A}{2}\right) \cos \left(\frac{\pi}{4} - \frac{B}{2}\right).$$

7. $\sin A - \sin B + \cos C - 1$
$$= -4 \sin \frac{C}{2} \sin \left(\frac{\pi}{4} - \frac{A}{2}\right) \cos \left(\frac{\pi}{4} - \frac{B}{2}\right).$$

8. $\sin (A - 60°) + \sin (B - 60°) + \sin (C - 60°)$
$$= -4 \sin \left(\frac{A}{2} - 30°\right) \sin \left(\frac{B}{2} - 30°\right) \sin \left(\frac{C}{2} - 30°\right).$$

9. $\sin (A + 2B - C) + \sin (2A - B + C) + \sin (-A + B + 2C)$
$$= 4 \cos \left(A - \frac{C}{2}\right) \cos \left(C - \frac{B}{2}\right) \cos \left(B - \frac{A}{2}\right).$$

10. $\sin (y - z) + \sin (z - x) + \sin (x - y)$
$$+ 4 \sin \frac{y - z}{2} \sin \frac{z - x}{2} \sin \frac{x - y}{2} = 0.$$

11. $\cos (y - z) + \cos (z - x) + \cos (x - y) + 1$
$$= 4 \cos \frac{y - z}{2} \cos \frac{z - x}{2} \cos \frac{x - y}{2}.$$

12. $4 \sin \alpha \sin \beta \sin \gamma$
$= \sin(\beta+\gamma-\alpha) + \sin(\gamma+\alpha-\beta) + \sin(\alpha+\beta-\gamma) - \sin(\alpha+\beta+\gamma).$

13. $4 \cos \alpha \cos \beta \sin \gamma$
$= \sin(\alpha+\beta+\gamma) + \sin(\beta+\gamma-\alpha) + \sin(\gamma+\alpha-\beta) - \sin(\alpha+\beta-\gamma).$

14. $4 \cos \alpha \cos \beta \cos \gamma$
$= \cos(\beta+\gamma-\alpha) + \cos(\gamma+\alpha-\beta) + \cos(\alpha+\beta-\gamma) + \cos(\alpha+\beta+\gamma).$

15. $4 \sin \alpha \sin \beta \cos \gamma$
$= \cos(\beta+\gamma-\alpha) + \cos(\gamma+\alpha-\beta) - \cos(\alpha+\beta-\gamma) - \cos(\alpha+\beta+\gamma).$

16. $\sin A(\cos B + \cos C) + \sin B(\cos C + \cos A)$
$+ \sin C(\cos A + \cos B) = \sin A + \sin B + \sin C.$

17. $\cot A + \dfrac{\sin A}{\sin B \sin C} = \cot C + \dfrac{\sin C}{\sin A \sin B}.$

18. $\dfrac{\sin A - \sin B \cos C}{\cos B} = \dfrac{\sin B - \sin A \cos C}{\cos A}.$

19. $\sin^2 A \sin 2C + \sin^2 C \sin 2A$
$= \sin^2 B \sin 2A + \sin^2 A \sin 2B.$

20. $\sin A \sin B + \cos^2\left(A + \dfrac{C}{2}\right) = \cos^2 \dfrac{C}{2}.$

21. $\sin A \sin(A + 2C) + \sin B \sin(B + 2A)$
$+ \sin C \sin(C + 2B) = 0.$

22. If $\cos A = \cos B \cos C$, then $\cot B \cot C = \frac{1}{2}$.

23. If $\sin A = \cos B \cos C$, then $\tan B + \tan C = 1$.

24. $\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2} = \cot \dfrac{A}{2} \cot \dfrac{B}{2} \cot \dfrac{C}{2}.$

25. $\cos \dfrac{A}{2} + \cos \dfrac{B}{2} + \cos \dfrac{C}{2} = 4 \cos \dfrac{\pi - A}{4} \cos \dfrac{\pi - B}{4} \cos \dfrac{\pi - C}{4}.$

26. $\sin A \sin B + \sin B \sin C + \sin C \sin A$
$= 2\left\{\cos \dfrac{B-C}{2} \cos \dfrac{C-A}{2} \cos \dfrac{A-B}{2} + \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}\right\}.$

27. $\cos A \cos B + \cos B \cos C + \cos C \cos A + 1$

$$= 2\left\{\cos\frac{B-C}{2}\cos\frac{C-A}{2}\cos\frac{A-B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right\}.$$

28. $\cos^2 x + \cos^2 y + \cos^2 z + 2\cos x\cos y\cos z - 1$

$$= 4\cos\frac{x+y+z}{2}\cos\frac{y+z-x}{2}\cos\frac{z+x-y}{2}\cos\frac{x+y-z}{2}.$$

29. $\sin^3 A + \sin^3 B + \sin^3 C$

$$= 3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}.$$

30. $\cos 3A\sin(B-C) + \cos 3B\sin(C-A)$

$+ \cos 3C\sin(A-B) = -4\sin(B-C)\sin(C-A)\sin(A-B)$.

[We have

$$\begin{aligned}2\cos 3A\sin(B-C) &= \sin(B-C+3A) - \sin(3A-B+C)\\ &= \sin(\pi+2A-2C) - \sin(\pi+2A-2B)\\ &= \sin(2C-2A) + \sin(2A-2B),\end{aligned}$$

similar expressions being found for the other two terms.

Thus

$$\begin{aligned}2\cos 3A\sin(B-C) + 2\cos 3B\sin(C-A) + 2\cos 3C\sin(A-B)&\\ = 2\{\sin(2A-2B) + \sin(2B-2C) + \sin(2C-2A)\}&\\ = -8\sin(B-C)\sin(C-A)\sin(A-B),&\end{aligned}$$ by Ex. 10.]

31. $\sin(B-C)\cos^3 A + \sin(C-A)\cos^3 B$

$+ \sin(A-B)\cos^3 C = -\sin(B-C)\sin(C-A)\sin(A-B)$.

[Substitute for $\cos^3 A$ in terms of $\cos 3A$ and $\cos A$.]

32. $\sin x\sin(y-z)\sin(y+z-x)$

$+ \sin y\sin(z-x)\sin(z+x-y) + \sin z\sin(x-y)\sin(x+y-z)$

$$= 2\sin(y-z)\sin(z-x)\sin(x-y).$$

[Using the results of Ex. 12, we have

$\sin x\sin(y-z)\sin(y+z-x)$

$$= \tfrac{1}{4}\{\sin 2(y-x) + \sin 2z + \sin 2(x-z) - \sin 2y\},$$

$\sin y \sin (z-x) \sin (z+x-y)$

$$= \tfrac{1}{4} \{\sin 2(z-y) + \sin 2x + \sin 2(y-x) - \sin 2z\},$$

$\sin z \sin (x-y) \sin (x+y-z)$

$$= \tfrac{1}{4} \{\sin 2(x-z) + \sin 2y + \sin 2(z-y) - \sin 2x\}.$$

Adding

$\sin x \sin (y-z) \sin (y+z-x) + \ldots\ldots$

$= \tfrac{1}{2} \{\sin 2(y-x) + \sin 2(z-y) + \sin 2(x-z)\}$

$= 2 \sin (y-z) \sin (z-x) \sin (x-y)$, by Ex. 10.]

33. $\cos x \sin (y-z) \cos (y+z-x)$

$+ \cos y \sin (z-x) \cos (z+x-y) + \cos z \sin (x-y) \cos (x+y-z)$

$$= 2 \sin (y-z) \sin (z-x) \sin (x-y).$$

34. $\sin y \sin z \sin (y-z) + \sin z \sin x \sin (z-x)$

$+ \sin x \sin y \sin (x-y) = - \sin (y-z) \sin (z-x) \sin (x-y).$

35. $\cos y \cos z \sin (y-z) + \cos z \cos x \sin (z-x)$

$+ \cos x \cos y \sin (x-y) = - \sin (y-z) \sin (z-x) \sin (x-y).$

36. If $\sin^3 \theta = \sin (\alpha - \theta) \sin (\beta - \theta) \sin (\gamma - \theta)$, where

$$\alpha + \beta + \gamma = \pi,$$

then $$\cot \theta = \cot \alpha + \cot \beta + \cot \gamma.$$

37. $\sin^4 A + \sin^4 B + \sin^4 C$

$$= \tfrac{1}{2} \{3 + 4 \cos A \cos B \cos C + \cos 2A \cos 2B \cos 2C\}.$$

38. $\cos^4 A + \cos^4 B + \cos^4 C$

$$= \tfrac{1}{2} \{1 - 4 \cos A \cos B \cos C + \cos 2A \cos 2B \cos 2C\}.$$

39. If $\tan \frac{3A}{4} \tan \frac{3B}{4} \tan \frac{3C}{4} = 1, \quad A + B + C = \pi,$

then

$$\tan \frac{3A}{4} + \tan \frac{3B}{4} + \tan \frac{3C}{4} = \cot \frac{3A}{4} + \cot \frac{3B}{4} + \cot \frac{3C}{4}.$$

40. If $\alpha + \beta + \gamma = \frac{\pi}{2}$

$$\frac{\left(1 - \tan\frac{\alpha}{2}\right)\left(1 - \tan\frac{\beta}{2}\right)\left(1 - \tan\frac{\gamma}{2}\right)}{\left(1 + \tan\frac{\alpha}{2}\right)\left(1 + \tan\frac{\beta}{2}\right)\left(1 + \tan\frac{\gamma}{2}\right)} = \frac{\sin\alpha + \sin\beta + \sin\gamma - 1}{\cos\alpha + \cos\beta + \cos\gamma}.$$

41. Show that any formula which is true when

$$A + B + C = \pi,$$

will also hold when for A, B, C are substituted (1) $\frac{\pi - A}{2}$, $\frac{\pi - B}{2}$, $\frac{\pi - C}{2}$; or (2) $\pi - 2A$, $\pi - 2B$, $\pi - 2C$; or (3) $2A - \frac{\pi}{3}$, $2B - \frac{\pi}{3}$, $2C - \frac{\pi}{3}$.

CHAPTER X*.

THE TRIGONOMETRICAL RATIOS OF SUBMULTIPLE ANGLES.

75. Having given $\cos\theta$ but not θ, $\cos\frac{\theta}{2}$ has two values.

We have already seen that

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1,$$

hence
$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2},$$

from which it follows that $\cos\frac{\theta}{2}$ either equals $+\sqrt{\frac{1+\cos\theta}{2}}$, or equals $-\sqrt{\frac{1+\cos\theta}{2}}$.

To know which sign to take, requires a knowledge of θ as well as $\cos\theta$. If θ be known, we know of what quadrant $\cos\frac{\theta}{2}$ is, and therefore its sign, and hence the sign to be prefixed to the radical.

The reason for the ambiguity which occurs when $\cos\theta$ but not θ is known, will be seen from the following considerations:

Suppose we are given that $\cos\theta = a$, a known quantity, also let α be the smallest angle whose cosine is a; then we saw in Article 43 that

$$\theta = 2n\pi \pm \alpha,$$

where n may be any integer.

Thus to have $\cos\theta$ equal to a given number, implies that θ belongs to a set of angles infinite in number.

Also,

$$\cos\frac{\theta}{2} = \cos\left(n\pi \pm \frac{\alpha}{2}\right) = \cos n\pi \cos\frac{\alpha}{2} \pm \sin n\pi \sin\frac{\alpha}{2},$$

but $$\sin n\pi = 0,$$

and $$\cos n\pi = \pm 1, \text{ according as } n \text{ is even or odd,}$$

hence $$\cos\frac{\theta}{2} = \pm\cos\frac{\alpha}{2}.$$

In like manner since

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{1},$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}.$$

And we cannot say which sign should be taken unless θ is known as well as $\cos\theta$.

Ex. When $\theta = 184°$, find $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$, having given

$$\cos 184° = -\cdot 9975641.$$

Here $\frac{\theta}{2} = 92°$ and is therefore of the second quadrant, hence its sine is positive, its cosine negative.

Thus $$\cos 92° = -\sqrt{\frac{1+\cos 184°}{2}} = -\sqrt{\frac{\cdot 0024359}{2}},$$

$$\sin 92° = +\sqrt{\frac{1-\cos 184°}{2}} = +\sqrt{\frac{1\cdot 9975641}{2}}.$$

76. When sin A is given, sin $\frac{A}{2}$ may be any one of four quantities.

We have

$$\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)^2=1+2\sin\frac{A}{2}\cos\frac{A}{2}=1+\sin A,$$

$$\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)^2=1-2\sin\frac{A}{2}\cos\frac{A}{2}=1-\sin A.$$

Extracting the square root

$$\cos\frac{A}{2}+\sin\frac{A}{2}=\pm\sqrt{1+\sin A}\ \ldots\ldots\ldots\ldots(1),$$

$$\cos\frac{A}{2}-\sin\frac{A}{2}=\pm\sqrt{1-\sin A}\ \ldots\ldots\ldots\ldots(2).$$

In each equation the radical may have either sign, this gives 4 cases, according as we take the signs of (1) and (2) to be

$$++,\quad +-,\quad --,\quad -+.$$

Take the first case of $++$, here (1) and (2) are,

$$\cos\frac{A}{2}+\sin\frac{A}{2}=+\sqrt{1+\sin A},\qquad \cos\frac{A}{2}-\sin\frac{A}{2}=+\sqrt{1-\sin A},$$

whence by addition we get

$$2\cos\frac{A}{2}=+\sqrt{1+\sin A}+\sqrt{1-\sin A},$$

and by subtraction

$$2\sin\frac{A}{2}=\sqrt{1+\sin A}-\sqrt{1-\sin A}.$$

The other three cases give three other values for $\cos\frac{A}{2}$, and for $\sin\frac{A}{2}$.

Thus when $\sin A$ is known, *but A not known*, there are 4 values which $\sin \frac{A}{2}$ may have, and 4 values which $\cos \frac{A}{2}$ may have.

77. When A is known, as well as $\sin A$, we can find what signs must be taken as follows:

$$\pm\sqrt{1 + \sin A} = \cos \frac{A}{2} + \sin \frac{A}{2}, \text{ from (1)},$$

$$\sqrt{2} \sin \left(\frac{A}{2} + 45^\circ\right) = \cos \frac{A}{2} + \sin \frac{A}{2}, \text{ by Article 54},$$

$$\therefore \quad \sqrt{2} \sin \left(\frac{A}{2} + 45^\circ\right) = \pm \sqrt{1 + \sin A},$$

also

$$\pm\sqrt{1 - \sin A} = \cos \frac{A}{2} - \sin \frac{A}{2}, \text{ from (2)},$$

$$\sqrt{2} \cos \left(\frac{A}{2} + 45^\circ\right) = \cos \frac{A}{2} - \sin \frac{A}{2}, \text{ by Article 54},$$

$$\therefore \quad \sqrt{2} \cos \left(\frac{A}{2} + 45^\circ\right) = \pm \sqrt{1 - \sin A},$$

so that the radicals are $\sqrt{2}$ times the sine and cosine of $\frac{A}{2} + 45^\circ$.

Now if A be known, we know of what quadrant $\frac{A}{2} + 45^\circ$ is; and therefore the signs of its sine and cosine, and hence the sign to be prefixed to each radical.

Ex. If $A = 170^\circ$, find the sign of each radical and the value of $\sin \frac{A}{2}$, having given $\sin 170^\circ = \cdot 1736482$.

Here $\frac{A}{2} + 45^\circ = 85^\circ + 45^\circ = 130^\circ$, is of the second quadrant, and therefore its sine is +, its cosine −,

thus we take + – as the signs, or (1) and (2) are in this case,

$$\cos 85° + \sin 85° = +\sqrt{1 + \sin 170°} = +\sqrt{1 \cdot 1736482},$$

$$\cos 85° - \sin 85° = -\sqrt{1 - \sin 170°} = -\sqrt{\cdot 8263518},$$

and subtracting

$$2 \sin 85° = \sqrt{1 + \sin 170°} + \sqrt{1 - \sin 170°} = 1 \cdot 9923894.$$

78. When tan A is given, tan $\frac{A}{2}$ has one of two values.

In Article 55, it was shown that

$$\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$$

or $$\tan^2 \frac{A}{2} . \tan A + 2 \tan \frac{A}{2} - \tan A = 0.$$

If $\tan A$ be given, this is a quadratic equation to find $\tan \frac{A}{2}$. Solving this equation it is found that

$$\tan \frac{A}{2} = \frac{-1 \pm \sqrt{1 + \tan^2 A}}{\tan A} = (-1 \pm \sec A) \cot A.$$

EXAMPLES. XXI.

1. Determine the signs to be taken in the formulae

$$\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}, \qquad \sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}},$$

when A equals

(i) 80°. (ii) 100°. (iii) 390°. (iv) 1000°.

2. Determine the signs to be taken in the formula

$$2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A},$$

when A equals

(i) $100°$. (ii) $260°$. (iii) $450°$. (iv) $1890°$.

3. Determine the proper sign in

$$\tan\frac{A}{2} = \cot A\,(\pm \sec A - 1),$$

when A equals

(i) $10°$. (ii) $200°$. (iii) $300°$. (iv) $8000°$.

4. Show that the positive sign is to be taken in (1), when A lies between $-90°$ and $270°$.

5. Show that the positive sign is to be taken in (2), when A lies between $-270°$ and $90°$.

6. Having given $\sin 220° = -\cdot 6427876$,

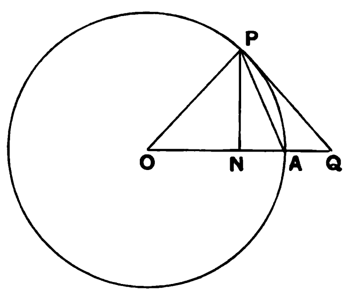
show that $\sin 110° = \cdot 9396926$, $\cos 110° = -\cdot 3420201$.

CHAPTER XI.

RELATIONS BETWEEN THE TRIGONOMETRICAL RATIOS AND THE CIRCULAR MEASURE OF AN ANGLE.

79. To show that when $\theta < \frac{\pi}{2}$, $\sin\theta < \theta < \tan\theta$.

If θ is the circular measure of an angle less than $\frac{\pi}{2}$, we shall show that $\sin\theta$ is less than θ, and θ is less than $\tan\theta$.



Let AOP be the angle θ, with O as centre and OA as radius, describe a circle cutting OP in P. Draw PN perpendicular to OA, and PQ perpendicular to OP. Join AP.

Then the $\triangle\, OAP$, the sector OAP, and the $\triangle\, OQP$ are seen to be in ascending order of magnitude. Also

$$\text{area of } \triangle\, OAP = \tfrac{1}{2}OA\,.\,PN = \tfrac{1}{2}OA^2\sin\theta,$$

since $$PN = OP \sin\theta = OA \sin\theta.$$

Area of sector $OAP = \frac{1}{2}OA^2 . \theta$, by Article 13.

Area of $\triangle OQP = \frac{1}{2}OP . PQ = \frac{1}{2}OP . OP \tan\theta = \frac{1}{2}OA^2 . \tan\theta$,

hence

$\frac{1}{2}OA^2 \sin\theta$ is $< \frac{1}{2}OA^2 . \theta$, which is itself $< \frac{1}{2}OA^2 \tan\theta$,

or omitting $\frac{1}{2}OA^2$,

$\sin\theta$ is $< \theta$, which is itself $< \tan\theta$.

80. When θ is indefinitely small,

$$\frac{\sin\theta}{\theta} = 1, \quad \frac{\tan\theta}{\theta} = 1.$$

From the inequality just proved, dividing throughout by $\sin\theta$, we get

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Hence $\frac{\theta}{\sin\theta}$ lies between unity and $\sec\theta$.

But when θ becomes indefinitely small, $\sec\theta$ becomes equal to unity, hence also must $\frac{\theta}{\sin\theta}$ become equal to unity.

Again $$\frac{\theta}{\tan\theta} = \frac{\theta}{\sin\theta} . \cos\theta.$$

If θ is indefinitely small $\frac{\theta}{\sin\theta} = 1$, and $\cos\theta = 1$,

$$\therefore \frac{\theta}{\tan\theta} = 1, \text{ if } \theta \text{ is indefinitely small.}$$

Thus if θ is a very small angle we may write θ in place of both $\sin\theta$ and $\tan\theta$ without appreciable error.

Notice that θ is *circular measure.* If there are n seconds in the angle whose circular measure is θ, we have

$$n = \frac{\theta}{\pi} \times 180 \times 60 \times 60, \text{ Article 11,}$$

and since $\sin n'' = \sin\theta$,

$$\frac{\sin n''}{n} = \frac{\sin\theta}{\theta} \times \frac{\pi}{180 \times 60 \times 60}$$

or $\quad \sin n'' = n \times \dfrac{\pi}{180 \times 60 \times 60}, \qquad$ since $\dfrac{\sin\theta}{\theta} = 1$;

and similarly

$$\tan n'' = n \times \frac{\pi}{180 \times 60 \times 60}.$$

81*. Limits for $\sin\theta$ and $\cos\theta$.

If θ is the circular measure of an angle less than $\frac{\pi}{2}$ we shall now show that

$$\sin\theta \text{ lies between } \theta \text{ and } \theta - \frac{\theta^3}{6},$$

$$\cos\theta \text{ lies between } 1 - \frac{\theta^2}{2} \text{ and } 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

By Article 70,

$$3\sin\frac{\theta}{3} - \sin\theta = 4\sin^3\frac{\theta}{3},$$

$$3\sin\frac{\theta}{3^2} - \sin\frac{\theta}{3} = 4\sin^3\frac{\theta}{3^2},$$

$$\dots\dots\dots\dots = \dots\dots\dots$$

$$3\sin\frac{\theta}{3^n} - \sin\frac{\theta}{3^{n-1}} = 4\sin^3\frac{\theta}{3^n}.$$

Multiply these equations by 1, 3, $3^2, \dots 3^{n-1}$ respectively and add, we have, since all the terms but two on the left side go out,

$$3^n\sin\frac{\theta}{3^n} - \sin\theta$$

$$= 4\left\{\sin^3\frac{\theta}{3} + 3\sin^3\frac{\theta}{3^2} + 3^2\sin^3\frac{\theta}{3^3} + \dots + 3^{n-1}\sin^3\frac{\theta}{3^n}\right\}$$

But we saw, in Article 79, that $\sin\alpha$ is $<\alpha$, hence

$$\sin^3\frac{\theta}{3} \text{ is } < \left(\frac{\theta}{3}\right)^3, \quad \sin^3\frac{\theta}{3^2} \text{ is } < \left(\frac{\theta}{3^2}\right)^3 \ldots\ldots$$

Therefore

$$3^n\sin\frac{\theta}{3^n} - \sin\theta$$

$$< 4\left\{\left(\frac{\theta}{3}\right)^3 + 3\left(\frac{\theta}{3^2}\right)^3 + 3^2\left(\frac{\theta}{3^3}\right)^3 + \ldots + 3^{n-1}\left(\frac{\theta}{3^n}\right)^3\right\}$$

or $$3^n\sin\frac{\theta}{3^n} - \sin\theta \text{ is } < \frac{4\theta^3}{3^3}\left\{1 + \frac{1}{3^2} + \frac{1}{3^4} + \ldots + \frac{1}{3^{2n-2}}\right\},$$

and $$1 + \frac{1}{3^2} + \frac{1}{3^4} + \ldots + \frac{1}{3^{2n-2}} = \frac{1-\left(\frac{1}{3^2}\right)^n}{1-\frac{1}{3^2}}.$$

thus, $$3^n\sin\frac{\theta}{3^n} - \sin\theta \text{ is } < \frac{4\theta^3}{3^3}\cdot\frac{1-\left(\frac{1}{3^2}\right)^n}{1-\frac{1}{3^2}}.$$

Now let n be taken to be indefinitely great, we may then write $\frac{\theta}{3^n}$ for $\sin\frac{\theta}{3^n}$, and θ for $3^n\sin\frac{\theta}{3^n}$, also $\left(\frac{1}{3^2}\right)^n$ becomes infinitely small,

hence $$\theta - \sin\theta \text{ is } < \frac{4\theta^3}{3^3}\,\frac{1}{1-\frac{1}{3^2}} \text{ or } < \frac{\theta^3}{6}.$$

Thus

$$\theta - \sin\theta \text{ is } < \frac{\theta^3}{6}, \text{ that is to say, } \sin\theta \text{ is } > \theta - \frac{\theta^3}{6}.$$

But we saw that $\sin\theta$ is $<\theta$, hence

$$\sin\theta \text{ lies between } \theta \text{ and } \theta - \frac{\theta^3}{6}.$$

Again

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}, \text{ and } \sin\frac{\theta}{2} \text{ is } < \frac{\theta}{2},$$

$$\therefore \cos\theta > 1 - 2\left(\frac{\theta}{2}\right)^2 \text{ or } > 1 - \frac{\theta^2}{2}.$$

Also since
$$\sin\frac{\theta}{2} \text{ is } > \frac{\theta}{2} - \frac{\left(\frac{\theta}{2}\right)^3}{6},$$

$$\cos\theta < 1 - 2\left\{\frac{\theta}{2} - \frac{\left(\frac{\theta}{2}\right)^3}{6}\right\}^2$$

or
$$< 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24};$$

hence $\cos\theta$ lies between

$$1 - \frac{\theta^2}{2} \text{ and } 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

82*. Euler's Product.

Since
$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

$$\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2^2}\cos\frac{\theta}{2^2},$$

$$\sin\frac{\theta}{2^2} = 2\sin\frac{\theta}{2^3}\cos\frac{\theta}{2^3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sin\frac{\theta}{2^{n-1}} = 2\sin\frac{\theta}{2^n}\cos\frac{\theta}{2^n}.$$

By equating the product of all the terms on the left to the product of all the terms on the right, we obtain

$$\sin\theta \,.\, \sin\frac{\theta}{2}\sin\frac{\theta}{2^2}\cdots\cdots\sin\frac{\theta}{2^{n-1}}$$

$$= 2^n \sin\frac{\theta}{2}\sin\frac{\theta}{2^2}\cdots\cdots\sin\frac{\theta}{2^{n-1}}\sin\frac{\theta}{2^n}\cos\frac{\theta}{2}\cdots\cdots\cos\frac{\theta}{2^n}$$

or $$\sin\theta = 2^n \sin\frac{\theta}{2^n}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\ldots\ldots\cos\frac{\theta}{2^n}.$$

Now increase n indefinitely, then

$$2^n \sin\frac{\theta}{2^n} = \theta,$$

and we have

$$\frac{\sin\theta}{\theta} = \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots\ldots \text{ to infinity}$$

which is known as Euler's product.

83*. *As θ increases from 0 to $\frac{\pi}{2}$, $\frac{\sin\theta}{\theta}$ continually diminishes, and $\frac{\tan\theta}{\theta}$ continually increases.*

For $$\theta + h \text{ being } > \theta \text{ and } < \frac{\pi}{2},$$

we shall show that,

$$\frac{\sin\theta}{\theta} > \frac{\sin(\theta+h)}{\theta+h},$$

that is $$(\theta+h)\sin\theta > \theta(\sin\theta\cos h + \cos\theta\sin h)$$

or $$\sin\theta\,(\theta + h - \theta\cos h) > \theta\cos\theta\sin h,$$

that is $$\frac{\tan\theta}{\theta} > \frac{\sin h}{h + \theta(1-\cos h)}.$$

Now we know that

$$\frac{\tan\theta}{\theta} > 1 > \frac{\sin h}{h},$$

and $$\frac{\sin h}{h} \text{ is } > \frac{\sin h}{h+\theta(1-\cos h)},$$

since $1 - \cos h$ is positive, hence

$$\frac{\tan\theta}{\theta} \text{ is greater than } \frac{\sin h}{h+\theta(1-\cos h)},$$

hence the inequality is proved: thus $\frac{\sin\theta}{\theta}$ diminishes from 1 to $\frac{2}{\pi}$, as θ increases from 0 to $\frac{\pi}{2}$.

Next we shall show that

$$\frac{\tan(\theta+h)}{\theta+h} > \frac{\tan\theta}{\theta},$$

or $$\theta\tan(\theta+h) > \overline{\theta+h}\tan\theta,$$

or that

$$\theta\sin(\theta+h)\cos\theta > (\theta+h)\sin\theta\cos(\theta+h),$$

or

$$\theta\{\sin(\theta+h)\cos\theta - \sin\theta\cos(\theta+h)\} > h\sin\theta\cos(\theta+h),$$

that is $$\theta\sin h > h\sin\theta\cos(\theta+h),$$

or $$\frac{\sin h}{h} > \frac{\sin\theta}{\theta}\cos(\theta+h).$$

We may suppose h is $< \theta$, hence by the first part

$$\frac{\sin h}{h} > \frac{\sin\theta}{\theta},$$

$$\text{hence } \frac{\sin h}{h} \text{ is greater than } \frac{\sin\theta}{\theta}\cos(\theta+h),$$

since $$\cos(\theta+h) < 1.$$

EXAMPLES. XXII.

1. Show that the limits of $\frac{\sin a\theta}{\theta}$ and $\frac{\sin p\theta}{\sin q\theta}$, are a and $\frac{p}{q}$, θ being indefinitely small.

2. Find the value of $m\sin\frac{a}{m}$ when m becomes indefinitely great.

3. In an isosceles triangle whose sides are a, a, $a \times 1{\cdot}022$, show that the base angles are $59^\circ\ 16'$ nearly.

4. From the top of a hill the depressions of two consecutive milestones are observed to be 3° and 5° respectively. Find the height of the hill.

5. A church whose distance is known subtends an angle of $10'$ at the eye, find its height approximately.

6. A man advancing on a level plane towards a vertical tower observes that at a certain point it subtends an angle of 2°, and after proceeding 1800 yards, in the same direction, an angle of 6°. Find approximately the height of the tower.

7. Two vertical rods on the Earth's surface each of which is 10·56 feet high ceases to be visible from the other when 8 miles distant. Show that the Earth's radius is 4000 miles nearly.

8. Show that the sine and cosine of an angle of n'' are approximately

$$\frac{n\pi}{648000}, \qquad 1 - \tfrac{1}{2}\left(\frac{n\pi}{648000}\right)^2.$$

9. Prove geometrically that $\tan\theta$ is $> 2\tan\dfrac{\theta}{2}$.

[See figure of Article 40.]

10. Prove that the value of the infinite product

$$\left(1-\tan^2\frac{\theta}{2}\right)\left(1-\tan^2\frac{\theta}{2^2}\right)\left(1-\tan^2\frac{\theta}{2^3}\right)\ldots\ldots \text{ is } \frac{\theta}{\tan\theta}.$$

11. Show that when θ is very small, the value of $\dfrac{1}{\sin^2\theta} - \dfrac{1}{\theta^2}$ is nearly $\tfrac{1}{3}$.

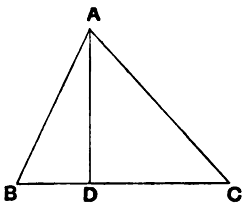
CHAPTER XII.

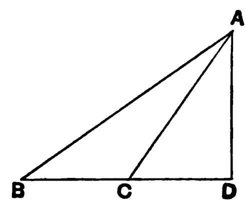
Relations between the Sides and Angles of a Triangle.

84. It has been already stated, in Chapter IV., that the angles of a triangle ABC are denoted by A, B and C; and the sides opposite to these angles by a, b and c.

The properties of triangles proved in the present chapter give the means by which a triangle can be solved, if three parts of it are given, one at least of those parts being a side.

85. The sides are proportional to the sines of the opposite angles.





In each figure AD is perpendicular to BC, and

$$\sin ABC = \frac{AD}{AB}, \quad \sin ACB = \frac{AD}{AC},$$

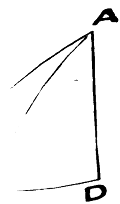
these angles
ual both to

n B we see

.........(1).

Chap. IV.

put equal to
cases.
last case.



if

In fig. ii. $a = BD - CD$

$$= AB \cos B - AC \cos ACD$$

and $\cos ACD = -\cos C$, since these angles are supplementary, therefore $a = c \cos B + b \cos C$.

By drawing perpendiculars from B and C two other similar results are obtained, we thus have

$$\left.\begin{aligned} a &= b \cos C + c \cos B \\ b &= c \cos A + a \cos C \\ c &= a \cos B + b \cos A \end{aligned}\right\} \text{..................(II).}$$

Ex. 1. The equations (II.) can be obtained from (I.) as follows:

$$\sin (B + C) = \sin (180° - A) = \sin A \qquad \text{Art. 44.}$$

$$\therefore \sin B \cos C + \sin C \cos B = \sin A.$$

Now let the value of each fraction in (I.) be called D, thus

$$a = D \sin A, \ b = D \sin B, \ c = D \sin C,$$

then the equation becomes

$$\frac{b}{D} \cos C + \frac{c}{D} \cos B = \frac{a}{D},$$

or

$$b \cos C + c \cos B = a.$$

Ex. 2. Show that

$$b^2 + c^2 - a^2 = 2bc \cos A.$$

Multiply the equations in (II.) by $-a$, b, and c respectively and add, we get

$$b^2 + c^2 - a^2 = 2bc \cos A.$$

This result will be proved independently.

Ex. 3. Prove that

$$\sin (B - C) = \frac{b^2 - c^2}{a^2} \sin A.$$

Replace a, b and c by their values as in Ex. 1, we have then to show that

$$\sin(B-C) = \frac{\sin^2 B - \sin^2 C}{\sin^2 A}\sin A,$$

since D^2 divides out. Now we have

$$\frac{\sin^2 B - \sin^2 C}{\sin^2 A} = \frac{(\sin B - \sin C)(\sin B + \sin C)}{\sin A \,.\, \sin(B+C)}$$

$$= \frac{4\cos\frac{B+C}{2}\sin\frac{B-C}{2}\,.\,\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\sin A\,.\,\sin(B+C)}$$

$$= \frac{2\sin\frac{B-C}{2}\cos\frac{B-C}{2}}{\sin A} = \frac{\sin(B-C)}{\sin A},$$

$$\therefore \qquad \frac{\sin^2 B - \sin^2 C}{\sin^2 A}\sin A = \sin(B-C).$$

Observe that in expressions involving only the *ratios* of a, b and c, we may replace them by $\sin A$, $\sin B$, and $\sin C$ respectively.

EXAMPLES. XXIII.

1. The angles of a triangle are 30°, 60°, 90°; the least side is 1 inch, show from (I.) that the other sides are $\sqrt{3}$ inches and 2 inches respectively.

2. If $a = \frac{\sqrt{6}-\sqrt{2}}{4}$, $b = \frac{1}{\sqrt{2}}$, $c = \frac{\sqrt{3}}{2}$ show that the angles are 15°, 45°, 120°.

Prove the following results

3. $\frac{a}{\sin A} = \frac{b+c}{\sin B + \sin C} = \frac{b-c}{\sin B - \sin C}$.

4. $\dfrac{a^3+b^3+c^3}{\sin^3 A+\sin^3 B+\sin^3 C}=\left(\dfrac{a}{\sin A}\right)^3.$

5. $a^{\frac{1}{2}}=(\sin A)^{\frac{1}{2}}\sqrt{\dfrac{a+2b-3c}{\sin A+2\sin B-3\sin C}}.$

6. If $a^2=bc$ show that

$$\cos(B-C)=1-\cos A-\cos 2A.$$

7. By putting $\sin(B+C)$ for $\sin A$ in Ex. 3, show that

$$\frac{a}{\sin\frac{A}{2}}=\frac{b+c}{\cos\frac{B-C}{2}},\quad \frac{a}{\cos\frac{A}{2}}=\frac{b-c}{\sin\frac{B-C}{2}}.$$

8. Prove that

$$b\sin B-c\sin C=a\sin(B-C).$$

Verify the following

9. $\dfrac{a-c\cos B}{b-c\cos A}=\dfrac{\sin B}{\sin A}.$

10. $\dfrac{a}{b}-\dfrac{b}{a}=c\left(\dfrac{\cos B}{b}-\dfrac{\cos A}{a}\right).$

11. $\dfrac{a}{\cos B}-\dfrac{b}{\cos A}=\cos C\left(\dfrac{b}{\cos B}-\dfrac{a}{\cos A}\right).$

12. If $a=2c$, $b=3c$, show that $\cos B=-1$.

13. In any triangle

$$(a+b)\cos C+c(\cos A+\cos B) \text{ is } > a\cos B+b\cos A.$$

14. $a\cos(B-C)+b\cos(C-A)+c\cos(A-B)$

$$+a\cos A+b\cos B+c\cos C=6a\sin B\sin C.$$

15. $2(a\cos A-b\cos B)\sin C=c(\sin 2A-\sin 2B).$

16. $c(\sin^2 A+\sin^2 B)=\sin C(a\sin A+b\sin B).$

17. $a\cos A+b\cos B+c\cos C=2a\sin B\sin C.$

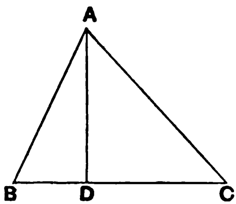
18. $a \sec A + b \sec B + c \sec C = a \sec A \tan B \tan C.$

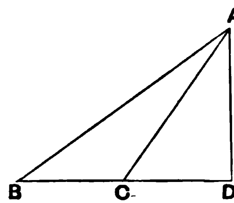
19. $4(a \sin A + b \sin B + c \sin C) \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

$$= (a + b + c)(\sin^2 A + \sin^2 B + \sin^2 C).$$

87. To prove that

$$c^2 = a^2 + b^2 - 2ab \cos C.$$





If AD be perpendicular to BC, we have when C is an acute angle (fig. 1)

$$AB^2 = AC^2 + BC^2 - 2BC \cdot CD, \qquad \text{Euc. II. 13.}$$

and when C is obtuse (fig. 2)

$$AB^2 = AC^2 + BC^2 + 2BC \cdot CD, \qquad \text{Euc. II. 12.}$$

In the first case

$$CD = AC \cos C,$$

and in the second case,

$$CD = AC \cos (180° - C) = - AC \cos C,$$

therefore in either case

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

By drawing perpendiculars from B and C we obtain two similar results, therefore

$$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= c^2 + a^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(\text{III}).$$

We may also write the equations as follows:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \text{ \&c.}$$

EXAMPLES. XXIV.

1. Given that $a = 4$, $b = 5$, $c = 6$, show that $\cos A = \frac{3}{4}$.

2. If $A = 60°$, $b = 8$, $c = 5$, prove that $a = 7$.

3. If the sides are as $2 : 3 : 4$, show that

$$\cos A : \cos B : \cos C = \tfrac{21}{6} : \tfrac{11}{4} : -1.$$

4. If $\dfrac{\cos A}{b} = \dfrac{\cos B}{a}$ the triangle is isosceles, or right angled.

5. If the sum of the squares of any two sides of a triangle is greater than the square of the third side, the triangle is acute angled.

6. $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$.

7. $\dfrac{b^2}{a} \cos A + \dfrac{c^2}{b} \cos B + \dfrac{a^2}{c} \cos C = \dfrac{a^4 + b^4 + c^4}{2abc}$.

8. By taking the vertical angle of an isosceles triangle as $2A$, prove by (III.) that $\cos 2A = 1 - 2\sin^2 A$.

9. The sides of a triangle are m, n, $\sqrt{m^2 + mn + n^2}$, show that the greatest angle is $120°$.

88. To show that

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

It is usual to denote the perimeter of a triangle by $2s$ so that $2s = a + b + c$; it follows that

$$2(s-a) = b + c - a,$$
$$2(s-b) = c + a - b,$$
$$2(s-c) = a + b - c.$$

We have

$$\sin^2\frac{A}{2} = \tfrac{1}{2}(1 - \cos A) \qquad \cos^2\frac{A}{2} = \tfrac{1}{2}(1 + \cos A),$$

and inserting for $\cos A$ its value $\dfrac{b^2+c^2-a^2}{2bc}$,

$$\sin^2\frac{A}{2} = \tfrac{1}{2}\left(1 - \frac{b^2+c^2-a^2}{2bc}\right), \quad \cos^2\frac{A}{2} = \tfrac{1}{2}\left(1 + \frac{b^2+c^2-a^2}{2bc}\right),$$

or

$$\sin^2\frac{A}{2} = \frac{a^2-(b-c)^2}{4bc}, \qquad \cos^2\frac{A}{2} = \frac{(b+c)^2-a^2}{4bc}$$

$$= \frac{(a-b+c)(a+b-c)}{4bc}, \qquad = \frac{(b+c+a)(b+c-a)}{4bc},$$

$$= \frac{4(s-b)(s-c)}{4bc}, \qquad = \frac{4s(s-a)}{4bc},$$

hence

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \qquad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \ldots(\text{IV}),$$

and by division

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

The values of $\sin\dfrac{B}{2}$, $\cos\dfrac{B}{2}$, $\tan\dfrac{B}{2}$ are got from these expressions by interchanging b and a, the values of

$$\sin\frac{C}{2},\ \cos\frac{C}{2},\ \tan\frac{C}{2},$$

by interchanging c and a. They should be written down as an exercise.

89. Area of a triangle.

In Art. 85, the area of the triangle is seen to be

$$\tfrac{1}{2} AD \,.\, BC = \tfrac{1}{2} c \sin B \,.\, a = \tfrac{1}{2} bc \sin A,$$

since $$b \sin A = a \sin B.$$

Denoting the area by S, we have

$$S = \tfrac{1}{2} bc \,.\, 2 \sin \frac{A}{2} \cos \frac{A}{2},$$

hence $$S = bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}},$$

$$S = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{..................(v).}$$

EXAMPLES. XXV.

1. If $a = 35$, $b = 84$, $c = 91$ show that

$$\tan \frac{A}{2} = \tfrac{1}{5}, \quad \tan \frac{B}{2} = \tfrac{2}{3}, \quad \tan \frac{C}{2} = 1, \quad S = 1470.$$

2. If $a = 125$, $b = 123$, $c = 62$, show that

$$\tan \frac{A}{2} = \tfrac{4}{5}, \quad \tan \frac{B}{2} = \tfrac{3}{4}, \quad \tan \frac{C}{2} = \tfrac{8}{31}, \quad S = 3720.$$

3. If $a = 13$, $b = 14$, $c = 15$,

$$\tan \frac{A}{2} = \tfrac{1}{2}, \quad \tan \frac{B}{2} = \tfrac{4}{7}, \quad \tan \frac{C}{2} = \tfrac{2}{3}, \quad S = 84.$$

4. If $a = 25$, $b = 52$, $c = 63$,

$$\tan \frac{A}{2} = \tfrac{1}{5}, \quad \tan \frac{B}{2} = \tfrac{1}{2}, \quad \tan \frac{C}{2} = \tfrac{9}{7}, \quad S = 630.$$

5. Show that the condition that $s - a$, $s - b$, $s - c$ may be proportional to the sides of some triangle is that s should be less than twice the smallest side.

6. If the condition in the last question is satisfied, show that if A', B', C' are the angles of this new triangle

$$\sin\frac{A'}{2}=\tfrac{1}{2}\sqrt{\frac{(2b-s)(2c-s)}{(s-b)(s-c)}},$$

$$\sin\frac{B'}{2}=\tfrac{1}{2}\sqrt{\frac{(2c-s)(2a-s)}{(s-c)(s-a)}},$$

$$\sin\frac{C'}{2}=\tfrac{1}{2}\sqrt{\frac{(2a-s)(2b-s)}{(s-a)(s-b)}}.$$

7. Prove that the following expressions are each equal to the area of the triangle

(1) $$\frac{a^2-b^2}{2}\,\frac{\sin A\sin B}{\sin(A-B)},$$

(2) $$s^2\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2},$$

(3) $$\frac{a^2+b^2-c^2}{4\tan\dfrac{A+B-C}{2}}.$$

90. To show that

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

This result, which is of use in the solution of triangles, is proved as follows:

By Art. 85. $$\frac{\sin B}{\sin C}=\frac{b}{c},$$

$$\therefore\ \frac{\sin B-\sin C}{\sin C}=\frac{b-c}{c},\quad \frac{\sin B+\sin C}{\sin C}=\frac{b+c}{c},$$

hence by division

$$\frac{\sin B-\sin C}{\sin B+\sin C}=\frac{b-c}{b+c},$$

also $$\frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2 \cos \frac{B+C}{2} \sin \frac{B-C}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}},$$

$$= \tan \frac{B-C}{2} \cot \frac{B+C}{2},$$

hence $$\frac{b-c}{b+c} = \tan \frac{B-C}{2} \cot \frac{B+C}{2},$$

$\therefore$ $$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \tan \frac{B+C}{2};$$

but since

$$\frac{B+C}{2} = 90^\circ - \frac{A}{2}, \quad \tan \frac{B+C}{2} = \cot \frac{A}{2},$$

or $$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

EXAMPLES. XXVI.

Prove the following results

1. $\frac{s(s-a)}{bc} - \frac{(s-b)(s-c)}{bc} = \cos A.$

2. $b \cos A - a \cos B = \frac{b^2 - a^2}{c}.$

3. $\frac{a}{\sin A} = \frac{a+b+c}{\sin A + \sin B + \sin C}.$

4. $\frac{a \cos B - b \cos A}{\sin (A-B)} = \frac{c}{\sin C}.$

5. $\frac{\cos A}{b} - \frac{\cos B}{a} = \frac{\cos C}{c} \left\{ \frac{\sin B}{\sin A} - \frac{\sin A}{\sin B} \right\}.$

6. $\cot C - \cot B = \frac{b^2 - c^2}{bc} \operatorname{cosec} A.$

7. $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0.$

8. $a \sin^2 C = c (\cos B + \cos A \cos C).$

9. $(s - b) \tan \dfrac{B}{2} = (s - c) \tan \dfrac{C}{2}.$

10. From (IV) show that $\sin \dfrac{A}{2} = \cos \dfrac{B + C}{2}.$

11. $a \cos A + b \cos B + c \cos C$

$$= \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{2abc}.$$

12. If $b = c$, and $A = 60°$, the triangle is equilateral.

13. If $b = 6$, $c = 4$, $\cos A = \frac{1}{3}$ then $a = 6$.

14. Given that $b = 15$, $c = 8$, $\cos A = \frac{1}{10}$; show that $a = 16 \cdot 3$ nearly.

15. If $b = 30$, $c = 20$, $\cos A = \frac{5}{6}$ prove that $a = 17 \cdot 3$ nearly.

16. If $B = 45°$, $C = 60°$, $a = 2(\sqrt{3} + 1)$ inches, show that $S = 6 + 2\sqrt{3}$ square inches.

17. Given that $a = 7$, $b = 8$, $c = 9$; show that the line joining B to the middle point of the opposite side $= 7$.

18. If $a = 17$ feet, $b = 25$ feet, $c = 26$ feet; $S = 204$ sq. feet.

19. If $a = 119$, $b = 111$, $c = 92$ yards, show that S is 10 sq. yards less than an acre.

20. If a^2, b^2, c^2 are in A.P., $a \sec A$, $b \sec B$, $c \sec C$ are in H.P.

21. Given that

$$2 \cos A + \cos B + \cos C = 2,$$

show that $$2a = b + c.$$

22. If A, B, C are in A.P.

$$2\cos\frac{A-C}{2}=\frac{a+c}{\sqrt{a^2-ac+c^2}}.$$

23. If a, b, c are in A.P.

$$\tan\frac{A}{2}\cdot\tan\frac{C}{2}=\tfrac{1}{3}.$$

If $A=30°$, $\cot\frac{B}{2}=4-\sqrt{3}$.

24. If D, E, F are the middle points of the sides

$$AD^2+BE^2+CF^2=\tfrac{3}{4}(a^2+b^2+c^2).$$

25. Show that the area and the trig. ratios of the triangle whose sides are

$$m^2+4n^2,\ 2m^2+2n^2,\ m^2+2n^2,$$

are all rational.

26. In any triangle if O is the orthocentre

$$\tan A=\frac{BC}{OA},\quad \tan B=\frac{CA}{OB},\quad \tan C=\frac{AB}{OC}.$$

27. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0.$

28. $b^2+c^2-2bc\cos(A+60°)=c^2+a^2-2ca\cos(B+60°)$

$$=a^2+b^2-2ab\cos(C+60°).$$

What is the geometrical meaning of this?

29. $a\sin A+b\sin B+c\sin C$

$$=2\{p\cos A+q\cos B+r\cos C\},$$

where p, q, r are the perpendiculars from the vertices on the opposite sides.

30. If $C=2B$, and A and B are not equal, then

$$c^2=ab+b^2.$$

31. The sides of a triangle are as

$$a+b,\ a-b,\ \sqrt{2(a^2+b^2)},$$

the sine of one angle is $\dfrac{\sqrt{5}-1}{4}$, find the other angles.

32. If $C=60^\circ$, $\dfrac{1}{a+c}+\dfrac{1}{b+c}=\dfrac{3}{a+b+c}$.

33. The sides of a triangle are in A.P. and the greatest angle exceeds the least by 90°; show that the sides are proportional to $\sqrt{7}+1,\ \sqrt{7},\ \sqrt{7}-1$.

34. Prove that

$$a\cos\frac{B-C}{2}=(b+c)\sin\frac{A}{2},$$

and

$$a\sin\frac{B-C}{2}=(b-c)\cos\frac{A}{2}.$$

CHAPTER XIII.

LOGARITHMS.

91. THE treatment of logarithms and their properties belongs to Algebra and is contained in all works on that subject. We shall merely state their chief properties for convenience of reference.

92. Definition of a logarithm.

If $x = a^m$, m is called the *logarithm* of x to the *base a*; this fact is expressed thus

$$m = \log_a x, \text{ or simply } m = \log x,$$

when it is understood what base is referred to.

In practical use of logarithms the base is always 10. The logarithms of all numbers from 1 to 100000, calculated to seven places of decimals, are contained in Chambers's Tables. The calculation to seven places of decimals gives results sufficiently accurate for ordinary purposes.

The fractional part of a logarithm is called the *mantissa*, the integral part the *characteristic*.

93. The chief properties of logarithms are the following:

$$\text{(i)} \quad \log_a mn = \log_a m + \log_a n.$$

$$\text{(ii)} \quad \log_a \frac{m}{n} = \log_a m - \log_a n.$$

$$\text{(iii)} \quad \log_a m^r = r \log_a m.$$

94. Rule of proportional differences.

It is proved in treatises on Algebra that $\log(1+h)$ is proportional to h, if h is any quantity so small that its square need not be considered.

[This follows from the fact that

$$\log_{10}(1+h) = A\left(h - \frac{h^2}{2} + \ldots\ldots\right)$$

where $A = \log_{10} e$, e being the number 2·718281828...]

Thus $\log(1+h) = A \,.\, h$, where A does not depend on h.

Therefore

$$\log(n+x) - \log n = \log\frac{n+x}{n} = \log\left(1 + \frac{x}{n}\right)$$

$$= A\frac{x}{n}, \text{ if } x \text{ is small compared with } n.$$

So that $\log(n+x) - \log n$ is proportional to x. This is expressed by saying that

the difference of two numbers and the difference of their logarithms are proportional; a fact known as the rule of *proportional differences.*

Taking another number a, also small compared with n, then in like manner

$$\log(n+a) - \log n = A\frac{a}{n};$$

and by division

$$\frac{\log(n+x) - \log n}{\log(n+a) - \log n} = \frac{A\frac{x}{n}}{A\frac{a}{n}}$$

or

$$\frac{\log(n+x) - \log n}{\log(n+a) - \log n} = \frac{x}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{I}).$$

Now assuming that n and $n+a$ are given numbers so that their logarithms are known, the formula I enables us

(i) if x is any given number smaller than a, to find $\log(n+x)$;

(ii) if $\log(n+x)$ is any given number between $\log(n+a)$ and $\log n$, to find x, and therefore the number $n+x$, of which it is the logarithm.

Ex. 1. Given $\log 581130 = 5{\cdot}7642733$,

$$\log 581140 = 5{\cdot}7642808,$$

find $$\log 581136.$$

Here
$$n = 581130,$$
$$a = 10,$$
$$x = 6.$$

From (I),

$$\log(n+x) = \log n + \frac{x}{a}\{\log(n+a) - \log a\}$$
$$= 5{\cdot}7642733 + \tfrac{6}{10}\{{\cdot}0000075\}$$
$$= 5{\cdot}7642733 + {\cdot}0000045$$
$$= 5{\cdot}7642778.$$

Ex. 2. Find the number whose logarithm is ${\cdot}6840700$, having given that $\log 4{\cdot}8313 = {\cdot}6840640$,

$$\log 4{\cdot}8314 = {\cdot}6840730.$$

We have to find x, given that

$$n = 4{\cdot}8313,$$
$$a = {\cdot}0001,$$
$$\log(n+x) = {\cdot}68407.$$

From (i),

$$x = a\frac{\log(n+x)-\log n}{\log(n+a)-\log n} = \cdot0001 \times \frac{\cdot000006}{\cdot000009} = \frac{6}{9} \times \cdot0001$$

$$= \cdot0000\dot{6}.$$

Required number $= n + x = 4\cdot8313\dot{6}$.

EXAMPLES. XXVII.

1. Find the logarithms of 1, 8, 128 to the base 16.

2. Given $\log 2 = \cdot3010300$, find $\log 625$ and $\log 2500$.

3. Find $\log_{\frac{1}{3}} 9$, $\log_{100} \cdot001$, $\log_2 \cos 45°$.

4. Given $\log 3 = \cdot4771213$, find $\log 75$, $\log \cdot00045$ and $\log \sqrt[3]{\cdot00012}$.

5. Find $\log 1944$, $\log \cdot003$, $\log \cdot00\dot{3}$.

6. Show that $\log 7 + \log 11 + \log 13$ is nearly 3.

7. Given $\log 104 = 2\cdot0170333$ find $\log 2\cdot6$.

8. Find the number of digits in the integral part of $(1\cdot04)^{1000}$, and the number of ciphers between the decimal point and the first significant figure of $(\cdot6)^{100}$.

9. If $\log \frac{1025}{1024} = p$ and $\log 2 = q$, then

$$\log 4100 = p + 12q.$$

10. Given

$$\log 59579 = 4\cdot7750933,$$
$$\log 5958 = 3\cdot7751005,$$

find

$$\log \cdot0595793.$$

11. Find the fifth root of $6\cdot4$, having given that

$$\log \cdot14495 = \bar{1}\cdot1612182,$$

$$\log \cdot14496 = \bar{1}\cdot1612482,$$

CHAPTER XIV.

TRIGONOMETRICAL TABLES.

95. To make the formulae of trigonometry of use in the solution of triangles, it is necessary that we should have *numerical tables* giving the trigonometrical ratios of angles, so that we may be able to find all the trigonometrical ratios of a given angle, and conversely the angle corresponding to a given ratio.

There are two kinds of such tables. The first being tables of "natural" sines, cosines &c., in which the numerical values of the sines &c., of angles are calculated to a certain number of places of decimals.

We give as a specimen a portion of such a table; the numbers have been taken from Chambers's Tables.

If sin 34° 17′ is required, looking for the number under sine and opposite 17 we find it to be ·5632857.

The numbers in the column headed *diff.* (or difference), are the changes in the sine, cosine, tangent and cotangent as the angle increases by 1′.

Thus

$$\sin 34^\circ\ 1' \text{ is } \textit{greater} \text{ than } \sin 34^\circ \text{ by } \cdot0002411;$$

$$\cos 34^\circ\ 1' \text{ is } \textit{less} \text{ than } \cos 34^\circ \text{ by } \cdot0001627.$$

34°

′	sine	diff.	cosine	diff.	tangent	diff.	cotangent	diff.
0	·5591929	2411	·8290376	1627	·6745085	4233	1·4825610	9299
1	·5594340	2411	·8288749	1628	·6749318	4235	1·4816311	9290
2	·5596751	2411	·8287121	1628	·6753553	4237	1·4807021	9283
3	·5599162	2410	·8285493	1629	·6757790	4238	1·4797738	9275
4	·5601572	2409	·8283864	1630	·6762028	4240	1·4788463	9266
5	·5603981	2409	·8282234	1631	·6766268	4241	1·4779197	9255
6	·5606390	2408	·8280603	1631	·6770509	4243	1·4769938	9250
7	·5608798	2408	·8278972	1632	·6774752	4245	1·4760688	9243
8	·5611206	2408	·8277340	1632	·6778997	4246	1·4751445	9235
9	·5613614	2407	·8275708	1634	·6783243	4249	1·4742210	9227
10	·5616021	2407	·8274074	1634	·6787492	4249	1·4732983	9219
11	·5618428	2406	·8272440	1634	·6791741	4252	1·4723764	9211
12	·5620834	2405	·8270806	1636	·6795993	4253	1·4714553	9203
13	·5623239	2406	·8269170	1636	·6800246	4255	1·4705350	9195
14	·5625645	2404	·8267534	1637	·6804501	4257	1·4696155	9188
15	·5628049	2404	·8265897	1637	·6808758	4258	1·4686967	9179
16	·5630453	2404	·8264260	1638	·6813016	4260	1·4677788	9172
17	·5632857	2403	·8262622	1639	·6817276	4261	1·4668616	9164
18	·5635260	2403	·8260983	1640	·6821537	4264	1·4659452	9156
19	·5637663	2403	·8259343	1640	·6825801	4265	1·4650296	9149
20	·5640066	2401	·8257703	1641	·6830066	4267	1·4641147	9140
21	·5642467	2402	·8256062	1642	·6834333	4268	1·4632007	9133
22	·5644869	2401	·8254420	1642	·6838601	4270	1·4622874	9125
23	·5647270		·8252778		·6842871		1·4613749	

96. Angles not exactly given in the table.

It may be required to find a trigonometrical ratio of an angle which lies between two consecutive angles of the table; for instance, sin 34° 17′ 45″. Its value will, of course, lie between the values given for sin 34° 17′ and sin 34° 18′.

To find its value we make use of the fact, proved later, that

a small change in the sine is proportional to the change in the angle.

So that if x is small (a certain number of seconds), $\sin(A+x)-\sin A$ is proportional to x, or,

$$\sin(A+x)-\sin A = K \,.\, x.$$

Also if B is nearly equal to A, or $B-A$ is small,

$$\sin B-\sin A = K(B-A),$$

whence by division,

$$\frac{\sin(A+x)-\sin A}{\sin B-\sin A} = \frac{x}{B-A}.$$

When B is taken $1'$, or $60''$, greater than A, then if A and B are angles exactly given in the table, $\sin B-\sin A$ is the *difference* of the table, so that

$$\sin(A+x)-\sin A = \frac{x}{60}\times \text{diff.} \ldots\ldots\ldots\ldots\ldots \text{I.}$$

From (I) if x be given, we can find $\sin(A+x)$ by use of the table; if $\sin(A+x)$ be given we can find x by using the table.

To find $\sin 34^\circ\ 17'\ 45''$.

Here $A = 34^\circ\ 17'$,

$x = 45$,

tabular difference $= \cdot 0002403$.

Therefore (I) gives us

$$\begin{aligned} \sin 34^\circ\ 17'\ 45'' &= \sin 34^\circ\ 17' + \tfrac{45}{60}\times \cdot 0002403 \\ &= \cdot 5632857 + \tfrac{3}{4}\times \cdot 0002403 \\ &= \cdot 5632857 + \cdot 0001802 \text{ nearly} \\ &= \cdot 5634659. \end{aligned}$$

97. In like manner, we get the equations

$$\tan(A+x)-\tan A = \frac{x}{60}\times \text{tabular difference} \ldots\ldots \text{II},$$

but $\cos A - \cos(A+x) = \frac{x}{60} \times$ tabular difference ....III,

and $\cot A - \cot(A+x) = \frac{x}{60} \times$ tabular difference ....IV.

N.B. The tabular difference multiplied by $\frac{x}{60}$ is to be *added to* $\sin A$, or $\tan A$, in order to get $\sin(A+x)$, or $\tan(A+x)$. But the tabular difference multiplied by $\frac{x}{60}$ is to be *subtracted from* $\cos A$, or $\cot A$, in order to get $\cos(A+x)$, or $\cot(A+x)$.

The equations I, II, III, and IV cannot be used, when A is very nearly 0° or 90° since it will be found that the reasoning on which they are founded does not hold in these cases.

Ex. 1. Find $\cot 34° 12' 42''$.

Here $A = 34° 12'$, $x = 42$,

$\cot 34° 12' = 1{\cdot}4714553$, tabular difference $= {\cdot}0009203$,

$\therefore$ by IV, $1{\cdot}4714553 - \cot(A+x) = \frac{42}{60} \times {\cdot}0009203$

or $$\cot(A+x) = 1{\cdot}4714553 - \tfrac{7}{10} \times {\cdot}0009203$$
$$= 1{\cdot}4708111.$$

Ex. 2. Find the angle whose tangent is ${\cdot}6808932$.

This number lies between $\tan 34° 15'$ and $\tan 34° 16'$.

Tabular difference $= {\cdot}0004258$.

$\therefore$ by II,

$$\cdot 6808932 - {\cdot}6808758 = \frac{x}{60} \times {\cdot}0004258$$

or $$x = \frac{{\cdot}6808932 - {\cdot}6808758}{{\cdot}0004258} \times 60 = 2{\cdot}4 \text{ nearly.}$$

The angle required is therefore

$$34^\circ\ 15'\ 2''{\cdot}4.$$

98*. To prove the rule of proportional differences for the sine.

We have that

$$\sin(A+x)-\sin A=\sin A(\cos x-1)+\cos A\sin x$$

$$=-\sin A\,.\,2\sin^2\frac{x}{2}+\cos A\sin x.$$

But when x is very small we may take $\frac{\pi x}{180}$ for $\sin x$ and $\frac{\pi x}{2\times 180}$ for $\sin\frac{x}{2}$, hence since x^2 may be neglected,

$$\sin(A+x)-\sin A=\frac{\pi x}{180}\cos A.$$

EXAMPLES. XXVIII.

Apply the methods just explained to find the following:

1. $\sin 34^\circ\ 16'\ 10''$.

2. $\cos 34^\circ\ 19'\ 32''$.

3. The angle whose cosine is $\cdot 8271356$.

4. $\sin 12^\circ\ 15'\ 20''$,

having given that

$$\sin 12^\circ\ 15'=\cdot 2121777,\quad \text{tabular diff.}=\cdot 0002842.$$

5. $\sin 40^\circ\ 38'\ 1''$,

having given that

$$\sin 40^\circ\ 38'=\cdot 6512158,\quad \text{diff.}=\cdot 0002208.$$

6. $\sin 53^\circ\ 39'\ 11''$,

$$\sin 53^\circ\ 39'=\cdot 8054113,\quad \text{tabular diff.}=\cdot 0001724.$$

7. $\tan 49^\circ\, 25'\, 16''$,

$\tan 49^\circ\, 25' = 1{\cdot}1674071$, diff. $= {\cdot}0006876$.

8. $\tan 70^\circ\, 51'\, 43''$,

$\tan 70^\circ\, 51' = 2{\cdot}8796979$, diff. $= {\cdot}0027054$.

9. $\tan 15^\circ\, 30'\, 45''$,

$\tan 15^\circ\, 30' = {\cdot}2775594$, diff. $= {\cdot}0003133$.

10. $\cos 12^\circ\, 21'\, 15''$,

$\cos 12^\circ\, 21' = {\cdot}9768593$, diff. $= {\cdot}0000623$.

11. $\cot 17^\circ\, 50'\, 10''$,

$\cot 17^\circ\, 50' = 3{\cdot}1084210$, diff. $= {\cdot}0030987$.

12. $\cos 42^\circ\, 59'\, 59''$,

$\cos 42^\circ\, 59' = {\cdot}7315521$, diff. $= {\cdot}0001984$.

13. $\cos 18^\circ\, 0'\, 1''$,

$\cos 18^\circ = {\cdot}9510565$, diff. $= {\cdot}0000899$.

14. $\cot 37^\circ\, 30'\, 4''$,

$\cot 37^\circ\, 30' = 1{\cdot}3032254$, diff. $= {\cdot}0007847$.

15. $\cot 77^\circ\, 10'\, 22''$,

$\cot 77^\circ\, 10' = {\cdot}2278063$, diff. $= {\cdot}0003060$.

16. Find the angle whose sine $= {\cdot}500005$,

$\sin 30^\circ = {\cdot}5$, diff. $= {\cdot}0002519$.

17. The angle whose sine $= {\cdot}3$,

$\sin 17^\circ\, 27' = {\cdot}2998734$, diff. $= {\cdot}0002775$.

18. The angle whose sine $= {\cdot}\dot{3}$,

$\sin 19^\circ\, 28' = {\cdot}3332584$, diff. $= {\cdot}0002742$.

19. The angle whose cosine $= {\cdot}9$,

$\cos 25^\circ\, 51' = {\cdot}9000654$, diff. $= {\cdot}0001268$.

20. The angle whose cotangent $= 2$,

$\cot 26^\circ\, 33' = 2{\cdot}0013142$, diff. $= {\cdot}0014552$.

21. The angle whose cosine $= \cdot 75$,

$\cos 41^\circ\ 24' = \cdot 7501111$, diff. $= \cdot 0001924$.

22. The angle whose cosine $= \cdot 125$,

$\cos 82^\circ\ 49' = \cdot 1250446$, diff. $= \cdot 0002886$.

23. The angle whose tangent $= 1 \cdot 3786523$,

$\tan 54^\circ\ 2' = 1 \cdot 3780672$, diff. $= \cdot 0008436$.

24. The angle whose tangent $= 4$,

$\tan 75^\circ\ 57' = 3 \cdot 9959223$, diff. $= \cdot 0049413$.

25. The angle whose tangent is 5.

$\tan 78^\circ\ 41' = 4 \cdot 9969459$, diff. $= \cdot 0075652$.

26. The angle whose cotangent $= 3 \cdot 6953285$.

$\cot 15^\circ\ 8' = 3 \cdot 6976104$, diff. $= \cdot 0042635$.

27. The angle whose cotangent $= \cdot 9578333$,

$\cot 46^\circ\ 14' = \cdot 9578494$, diff. $= \cdot 0005577$.

99. Tables of logarithmic sines.

The second kind of tables is that which gives the logarithms of the sine, cosine &c., of angles, or the *logarithmic sines, logarithmic cosines*, &c. as they are called.

Since the sine and cosine are always less than unity, and the tangent and cotangent *may be* less than unity, their logarithms are then negative. In order to avoid having negative logarithms it is usual to add 10 to the logarithm, which is then written $L \sin A$, $L \cos A$, &c., thus

$$L \sin A = 10 + \log \sin A.$$

These numbers are then called *tabular* logarithms.

We add a portion of a table of tabular logarithms for the sine, cosine, tangent and cotangent of angles between 25° and 25° 25′.

25°

′	L sine	diff.	L cosine	diff.	L tangent	diff.	L cotangent
0	9·6259483	2708	9·9572757	589	9·6686725	3298	10·3313275
1	9·6262191	2706	9·9572168	590	9·6690023	3296	10·3309977
2	9·6264897	2704	9·9571578	590	9·6693319	3294	10·3306681
3	9·6267601	2702	9·9570988	591	9·6696613	3293	10·3303387
4	9·6270303	2700	9·9570397	591	9·6699906	3291	10·3300094
5	9·6273003	2698	9·9569806	591	9·6703197	3289	10·3296803
6	9·6275701	2696	9·9569215	592	9·6706486	3288	10·3293514
7	9·6278397	2693	9·9568623	593	9·6709774	3286	10·3290226
8	9·6281090	2692	9·9568030	593	9·6713060	3285	10·3286940
9	9·6283782	2690	9·9567437	593	9·6713345	3283	10·3283655
10	9·6286472	2688	9·9566844	594	9·6719628	3282	10·3280372
11	9·6289160	2685	9·9566250	594	9·6722910	3280	10·3277090
12	9·6291845	2684	9·9565656	595	9·6726190	3278	10·3273810
13	9·6294529	2682	9·9565061	595	9·6729468	3277	10·3270532
14	9·6297211	2679	9·9564466	596	9·6732745	3275	10·3267255
15	9·6299890	2678	9·9563870	596	9·6736020	3274	10·3263980
16	9·6302568	2675	9·9563274	596	9·6739294	3272	10·3260706
17	9·6305243	2674	9·9562678	597	9·6742566	3270	10·3257434
18	9·6307917	2672	9·9562081	598	9·6745836	3269	10·3254164
19	9·6310589	2669	9·9561483	597	9·6749105	3267	10·3250895
20	9·6313258	2668	9·9560886	599	9·6752372	3266	10·3247628
21	9·6315926	2665	9·9560287	598	9·6755638	3265	10·3244362
22	9·6318591	2664	9·9559689	600	9·6758903	3262	10·3241097
23	9·6321255	2661	9·9559089	599	9·6762165	3261	10·3237835
24	9·6323916	2660	9·9558490	600	9·6765426	3260	10·3234574
25	9·6326576		9·9557890		9·6768686		10·3231314

Ex. To find $L \cot 25° \ 16'$. The table gives it as 10·3260706.

100. Logarithms not exactly given in the table.

As in the case of "natural" sines, if it is required to find the logarithmic sine of an angle not exactly given in the table we use the fact proved in Article 101, that

A small change in the logarithmic sine is proportional to the change in the angle itself. So that A being any angle contained in the table,

$$L \sin (A + x) - L \sin A = K \,.\, x,$$

x being a certain number of seconds, and K not depending on x.

Thus if B is greater than A by $60''$,

$$L\sin B - L\sin A = K \,.\, 60,$$

but $L\sin B - L\sin A$ is given as a difference in the table, hence by division,

$$L\sin(A+x) - L\sin A = \frac{x}{60} \times \text{tabular difference} \ldots \text{V}.$$

Whence if x is known we can find $L\sin(A+x)$, and if $L\sin(A+x)$ is given we can find x.

For the logarithmic tangents, cosines and cotangents we have

$$L\tan(A+x) - L\tan A = \frac{x}{60} \times \text{tabular diff.} \ldots\ldots \text{VI},$$

but $$L\cos A - L\cos(A+x) = \frac{x}{60} \times \text{tabular diff.} \ldots \text{VII},$$

and $$L\cot A - L\cot(A+x) = \frac{x}{60} \times \text{tabular diff.} \ldots \text{VIII}.$$

To get $L\sin(A+x)$, and $L\tan(A+x)$ we *add* $\frac{x}{60}$ times the tabular difference to $L\sin A$ or $L\tan A$; to get $L\cos(A+x)$ or $L\cot(A+x)$ we *subtract* $\frac{x}{60}$ times the tabular difference from $L\cos A$ or $L\cot A$.

For reasons which cannot be entered into fully in an elementary work, the above rules do not apply when the angle A is very nearly $0°$ or $90°$.

Ex. 1. Find $L\sin 25° \; 13' \; 12''$.

We have $A = 25° \; 13', \quad x = 12,$

$$L\sin A = 9{\cdot}6294529, \quad \text{tabular diff.} = {\cdot}0002682,$$

$\therefore$ by V,

$$L \sin 25^\circ\ 13'\ 12'' = 9{\cdot}6294529 + \tfrac{12}{60} \times {\cdot}0002682$$
$$= 9{\cdot}6295065.$$

Ex. 2. Find the angle whose tabular logarithmic sine is 9·6314276.

The table tells us that the angle lies between 25° 20′ and 25° 21′.

Thus $\quad A = 25^\circ\ 20'$, $\quad$ tabular diff. $= {\cdot}0002668$,

$$L \sin A = 9{\cdot}6313258,$$
$$L \sin (A + x) = 9{\cdot}6314276,$$

$\therefore$ by V,

$$9{\cdot}6314276 - 9{\cdot}6313268 = \frac{x}{60}\,({\cdot}0002668),$$

$$x = 60 \times \frac{{\cdot}0001008}{{\cdot}0002668} = 60 \times \frac{1008}{2668} = 22{\cdot}7 \text{ nearly.}$$

$$\text{Angle required} = 25^\circ\ 20'\ 22{\cdot}7''.$$

101*. To prove that a small change in the logarithmic sine is proportional to the change in the angle.

$$L \sin (A + x) - L \sin A = \log \sin (A + x) - \log \sin A$$
$$= \log \frac{\sin (A + x)}{\sin A} = \log (\cos x + \sin x \cot A)$$
$$= \log \cos x + \log (1 + \tan x \cot A).$$

Now when x is very small we may replace $\cos x$ by unity, and $\tan x$ by $\dfrac{\pi x}{180}$.

Also, see Art. 94,

$$\log \left(1 + \frac{\pi x}{180} \cot A\right) = \frac{\pi x}{180} \cot A \times \log_{10} e,$$

$$\therefore\ L \sin (A + x) - L \sin A = \frac{\pi x}{180} \cot A \times \log_{10} e.$$
$$= Kx,$$

where K is independent of x.

EXAMPLES. XXIX.

Find the value of the following:

1. $L\sin 13^\circ\ 31'\ 40''$,

having given

$L\sin 13^\circ\ 31' = 9{\cdot}3687111$, difference for $1' = {\cdot}0005252$.

2. $L\sin 27^\circ\ 48'\ 5''$,
   $L\sin 27^\circ\ 48' = 9{\cdot}6687461$, diff. $= {\cdot}0002395$.

3. $L\sin 30^\circ\ 0'\ 1''$,
   $L\sin 30^\circ = 9{\cdot}6989700$, diff. $= {\cdot}0002187$.

4. $L\cos 30^\circ\ 0'\ 1''$,
   $L\cos 30^\circ = 9{\cdot}9375306$, diff. $= {\cdot}0000729$.

5. $L\cos 44^\circ\ 59'\ 10''$,
   $L\cos 44^\circ\ 59' = 9{\cdot}8496113$, diff. $= {\cdot}0001263$.

6. $L\cos 63^\circ\ 18'\ 17''$,
   $L\cos 63^\circ\ 18' = 9{\cdot}6525548$, diff. $= {\cdot}0002513$.

7. $L\tan 10^\circ\ 30'\ 8''$,
   $L\tan 10^\circ\ 30' = 9{\cdot}2606330$, diff. $= {\cdot}0006811$.

8. $L\tan 75^\circ\ 10'\ 7''$,
   $L\tan 75^\circ\ 10' = 10{\cdot}5770265$, diff. $= {\cdot}0005107$.

9. $L\tan 77^\circ\ 1'\ 2''$,
   $L\tan 77^\circ\ 1' = 10{\cdot}6372126$, diff. $= {\cdot}0005774$.

10. $L\cot 18^\circ\ 35'\ 13''$,
    $L\cot 18^\circ\ 35' = 10{\cdot}4733850$, diff. $= {\cdot}0004181$.

11. $L\cot 17^\circ\ 25'\ 3''$,
    $L\cot 17^\circ\ 25' = 10{\cdot}5034848$, diff. $= {\cdot}0004422$.

12. $L \cot 60° \, 38' \, 47''$,

$L \cot 60° \, 38' = 9{\cdot}7502806$, diff. $= {\cdot}0002956$.

13. Find the angle whose log. sine is $9{\cdot}7169223$, given that

$L \sin 31° \, 24' = 9{\cdot}7168458$, diff. for $1' = {\cdot}0002068$.

14. Find the angle whose log. sine is $9{\cdot}1855273$,
$L \sin 8° \, 49' = 9{\cdot}1854665$, diff. $= {\cdot}0008137$.

15. Find the angle whose log. sine is $9{\cdot}9877113$,
$L \sin 76° \, 26' = 9{\cdot}9877099$, diff. $= {\cdot}0000305$.

16. Find the angle whose log. sine is $9{\cdot}5955550$,
$L \sin 23° \, 12' = 9{\cdot}5954322$, diff. $= {\cdot}0002946$.

17. Find the angle whose log. cosine is $9{\cdot}9900367$,
$L \cos 12° \, 13' = 9{\cdot}9900521$, diff. $= {\cdot}0000274$.

18. Find the angle whose log. cosine is $9{\cdot}7977635$,
$L \cos 51° \, 7' = 9{\cdot}7977775$, diff. $= {\cdot}0001567$.

19. Find the angle whose log. tan is $10{\cdot}4378605$,
$L \tan 69° \, 57' = 10{\cdot}4377561$, diff. $= {\cdot}0003924$.

20. Find the angle whose log. tan is $9{\cdot}7766638$,
$L \tan 30° \, 52' = 9{\cdot}7764816$, diff. $= {\cdot}0002869$.

CHAPTER XV.

SOLUTION OF TRIANGLES.

102. THE results of Chapters 12 and 14 are of use in enabling us to determine all the sides and angles of a triangle when three of them are known.

It is necessary that of the six parts of a triangle, viz. the three sides and three angles, one of the three given parts should be a side. There are thus four ways in which the parts may be given, which will now be considered.

103. CASE I.

Having given two angles and a side; as A, C, a.

Then B is determined from the equation

$$B = 180^\circ - A - C;$$

the sides b, c are found from

$$b = a\frac{\sin B}{\sin A}, \quad c = a\frac{\sin C}{\sin A} \qquad \text{Art. 85.}$$

or taking logarithms

$$\log b = \log a + L\sin B - L\sin A,$$

$$\log c = \log a + L\sin C - L\sin A.$$

Ex. If $a = 10$, $A = 51° 30' 40''$, $B = 76°$ find b, having given

$$L \sin 76° = 9{\cdot}9869041, \qquad \log 12{\cdot}3963 = 1{\cdot}0932928,$$
$$L \sin 51° 30' = 9{\cdot}8935444,$$
$$L \sin 51° 31' = 9{\cdot}8936448,$$

we have

$$\log b = \log 10 + L \sin 76° - L \sin 51° 30' 40'',$$

and $\quad L \sin 51° 30' 40'' = L \sin 51° 30' + \frac{40}{60} \times {\cdot}0001004$

$$= 9{\cdot}8936113,$$

hence $\quad \log b = 1 + 9{\cdot}9869041 - 9{\cdot}8936113$

$$= 1{\cdot}0932928,$$

therefore $\quad b = 12{\cdot}3963.$

EXAMPLES. XXX.

1. If $a = 15$, $A = 42°$, $B = 36°$ find b and c, having given,

$$L \sin 42° = 9{\cdot}8255109, \qquad \log 13{\cdot}17647 = 1{\cdot}1197991,$$
$$L \sin 36° = 9{\cdot}7692187, \qquad \log 21{\cdot}92728 = 1{\cdot}3409848,$$
$$L \sin 102° = 9{\cdot}9904044. \qquad \log 15 = 1{\cdot}1760913.$$

2. $a = 123$, $B = 29° 17'$, $C = 135°$ find c, given that

$$\log 123 = 2{\cdot}0899051,$$
$$\log 3211 = 4{\cdot}5066403, \text{ difference for } 1 = 1352,$$
$$\log 2 = {\cdot}3010300,$$
$$L \sin 15° 43' = 9{\cdot}4327777.$$

3. $a = 1$, $A = 49°$, $B = 51°$, find b and c: having given

$$L \sin 51° = 9{\cdot}8905026, \qquad \log 1{\cdot}029729 = {\cdot}0127227,$$
$$L \sin 49° = 9{\cdot}8777799, \qquad \log 1{\cdot}304883 = {\cdot}1155716,$$
$$L \sin 80° = 9{\cdot}9933515.$$

4. $a = 10$, $B = 80°$, $C = 60°$, find b and c.

$L \sin 80° = 9{\cdot}9933515$, $\log 1347296 = 6{\cdot}1294631$,
$L \sin 40° = 9{\cdot}8080675$, $\log 1532089 = 6{\cdot}1852840$.
$L \sin 60° = 9{\cdot}9375306$.

5. $a = 4$, $A = 15°$, $C = 95°$, find c.

$L \sin 15° = 9{\cdot}4129962$, $\log 15396 = 4{\cdot}1874079$,
$L \cos 5° = 9{\cdot}9983442$.

6. $a = 12$, $A = 25°$, $B = 35°$, find b and c.

$L \sin 35° = 9{\cdot}7585913$, $\log 12 = 1{\cdot}0791812$,
$L \sin 25° = 9{\cdot}6259483$, $\log 1628637 = 6{\cdot}2118242$,
$L \cos 30° = 9{\cdot}9375306$, $\log 2459028 = 6{\cdot}3907635$.

7. $a = 29$, $A = 81° \, 35'$, $B = 17° \, 58'$, solve the triangle.

$L \sin 81° \, 35' = 9{\cdot}9952972$, $\log 90428 = 4{\cdot}9563029$,
$L \sin 17° \, 58' = 9{\cdot}4892040$, $\log 90429 = 4{\cdot}9563077$,
$L \sin 80° \, 27' = 9{\cdot}9939391$, $\log 28909 = 4{\cdot}4610331$,
$\log 29 = 1{\cdot}4623980$, $\log 28910 = 4{\cdot}4610481$.

8. $a = 17$, $A = 53° \, 28'$, $B = 29°$.

$L \sin 53° \, 28' = 9{\cdot}9049916$, $\log 102571 = 5{\cdot}0110246$,
$L \sin 29° = 9{\cdot}6855712$, $\log 102572 = 5{\cdot}0110288$,
$L \cos 7° \, 32' = 9{\cdot}9962352$, $\log 20974 = 4{\cdot}3216813$,
$\log 17 = 1{\cdot}2304489$, $\log 20975 = 4{\cdot}3217020$.

9. $a = 7$, $A = 59°$, $B = 60°$.

$L \sin 59° = 9{\cdot}9330656$, $\log 70723 = 4{\cdot}8495607$,
$L \sin 60 = 9{\cdot}9375306$, $\log 70724 = 4{\cdot}8495668$,
$L \sin 61° = 9{\cdot}9418193$, $\log 71425 = 4{\cdot}8538502$,
$\log 7 = {\cdot}8450980$, $\log 71426 = 4{\cdot}8538563$.

10. $a = 101$, $A = 23°$, $B = 88°$.

$L \sin 88° = 9{\cdot}9997354$, $\log 25833 = 4{\cdot}4121748$,
$L \sin 23° = 9{\cdot}5918780$, $\log 25834 = 4{\cdot}4121917$,
$L \sin 69° = 9{\cdot}9701517$, $\log 24132 = 4{\cdot}3825933$,
$\log 101 = 2{\cdot}0043214$, $\log 24133 = 4{\cdot}3826113$.

104. Case II.

Having given the three sides, a, b, c.

The angles are determined by the formulae

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}},$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}},$$

or by

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}},$$

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}.$$

Thus we have

$$L\tan\frac{A}{2} = 10 + \tfrac{1}{2}\log\frac{(s-b)(s-c)}{s(s-a)}.$$

Ex. The sides of a triangle are proportional to 4, 7, 9; find the angles, having given

$$\log 2 = {\cdot}3010300,$$

$$L\tan 12°\,36' = 9{\cdot}349329, \text{ diff. for } 1' = {\cdot}000593,$$

$$L\tan 24°\,5' = 9{\cdot}650281, \text{ diff. for } 1' = {\cdot}000339.$$

We find $s=10$, $s-a=6$, $s-b=3$, $s-c=1$, hence

$$\tan\frac{A}{2}=\sqrt{\tfrac{1}{20}},\quad \tan\frac{B}{2}=\sqrt{\tfrac{2}{10}},$$

therefore $$L\tan\frac{A}{2}=10+\tfrac{1}{2}(-\log 20)$$
$$=10-\tfrac{1}{2}(1+\cdot3010300)$$
$$=9\cdot349485,$$

and $$L\tan\frac{B}{2}=10+\tfrac{1}{2}(\log 2-1)=9\cdot650515.$$

Thus $$\frac{A}{2}=12^\circ\,36'+\frac{9\cdot349485-9\cdot349329}{\cdot000593}\times 60''$$
$$=12^\circ\,36'+\frac{\cdot000156}{\cdot000593}\times 60''$$
$$=12^\circ\,36'\,15\cdot8'',$$

and $$\frac{B}{2}=24^\circ\,5'+\frac{9\cdot650515-9\cdot650281}{\cdot000339}\times 60''$$
$$=24^\circ\,5'\,41\cdot4''.$$

So that $A=25^\circ\,12'\,31\cdot6''$, $B=48^\circ\,11'\,22\cdot8''$,

hence $C=180^\circ-A-B=106^\circ\,36'\,5\cdot6''$.

EXAMPLES. XXXI.

1. $a=15$, $b=16$, $c=29$, find C given that
$L\tan 69^\circ\,17'=10\cdot4222774$, diff. for $1'=3819$.
$\log 7=\cdot8450980$.

2. $a=7$, $b=20$, $c=25$, find A, where,
$L\tan 6^\circ\,17'=9\cdot0418134$, diff. for $1'=11597$,
$\log 13=1\cdot1139434$,
$\log 19=1\cdot2787536$.

3. Find the greatest angle of a triangle whose sides are 183, 195 and 214 feet long.

$\log 82 = 1{\cdot}9138139$, $\log 296 = 2{\cdot}4712917$,

$\log 101 = 2{\cdot}0043214$, $L \tan 34^\circ\, 26' = 9{\cdot}8360513$,

$\log 113 = 2{\cdot}0530784$, $L \tan 34^\circ\, 27' = 9{\cdot}8363221$.

4. If $a = 5$, $b = 2$, $c = 6$, find all the angles.

$L \tan 25^\circ\, 39' = 9{\cdot}6814160$, diff. for $1' = 3236$.

$L \tan 9^\circ\, 5' = 9{\cdot}2037825$, diff. for $1' = 8097$.

$\log 13 = 1{\cdot}1139434$.

5. $a = 9$, $b = 10$, $c = 11$, find all the angles.

$L \tan 25^\circ\, 14' = 9{\cdot}6732745$, diff. for $1' = 3275$,

$L \tan 29^\circ\, 29' = 9{\cdot}7523472$, diff. for $1' = 2948$.

$\log 3 = {\cdot}4771213$.

6. $a = 45$, $b = 175$, $c = 210$, find the angles.

$L \tan 4^\circ\, 13' = 8{\cdot}8676317$, diff. for $1' = 17194$.

$L \tan 17^\circ\, 27' = 9{\cdot}4973991$, diff. for $1' = 4415$.

$\log 43 = 1{\cdot}6334685$, $\log 17 = 1{\cdot}2304489$.

7. $a = 3$, $b = 9$, $c = 8$, find the angles.

$L \tan 9^\circ\, 35' = 9{\cdot}2274706$, diff. for $1' = 7689$,

$L \tan 49^\circ\, 47' = 10{\cdot}0728534$, diff. for $1' = 2562$.

8. $a = 222$, $b = 318$, $c = 406$.

$\log 473 = 2{\cdot}6748611$, $\log 155 = 2{\cdot}1903317$,

$\log 251 = 2{\cdot}3996737$, $\log 222 = 2{\cdot}3463530$,

$\log 318 = 2{\cdot}5024271$, $L \cos 25^\circ\, 35' = 9{\cdot}9551864$,

$\log 406 = 2{\cdot}6085260$, $L \cos 25^\circ\, 36' = 9{\cdot}9551259$,

$L \cos 16^\circ\, 28' = 9{\cdot}9818117$,

$L \cos 16^\circ\, 29' = 9{\cdot}9817744$.

9. $a = 13$, $b = 14$, $c = 15$.

$L \tan 26° 33' = 9{\cdot}6986847$, tabular diff. for $1' = 3159$,
$L \tan 29° 44' = 9{\cdot}7567587$, tabular diff. for $1' = 2933$,
$L \tan 33° 41' = 9{\cdot}8237981$, tabular diff. for $1' = 2738$.

105. Case III.

Having given two sides and the included angle as b, c, A.

B and C may be determined from

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}, \qquad \text{Art. 90.}$$

together with $\quad B + C = 180° - A.$

Taking logarithms, we have

$$L\tan\frac{B-C}{2} = \log(b-c) - \log(b+c) + L\cot\frac{A}{2}.$$

Having found B and C the side a may be found from any one of the formulae

$$\log a = \log c + L\sin A - L\sin C,$$

$$\log a + L\cos\frac{B-C}{2} = \log(b+c) + L\sin\frac{A}{2},$$

$$\log a + L\sin\frac{B-C}{2} = \log(b-c) + L\cos\frac{A}{2}. \quad \text{See Ex. 34, p. 149.}$$

We may also determine a as follows:

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

This formula is not adapted for use with logarithms, but we may write it

$$a^2 = (b+c)^2 - 4bc\cos^2\frac{A}{2},$$

now take an angle ϕ such that

$$\sin\phi = \frac{2\sqrt{bc}\cos\dfrac{A}{2}}{b+c},$$

then we have

$$a^2 = (b+c)^2 \left\{1 - \frac{4bc}{(b+c)^2}\cos^2\frac{A}{2}\right\},$$

$$= (b+c)^2 (1 - \sin^2\phi),$$

$$= (b+c)^2 \cos^2\phi,$$

so that $\qquad a = (b+c)\cos\phi.$

Thus we may find ϕ by the logarithmic formula

$$L\sin\phi = \log 2 + \tfrac{1}{2}\log b + \tfrac{1}{2}\log c + L\cos\frac{A}{2} - \log(b+c),$$

and then determine a by the formula

$$\log a = \log(b+c) + L\cos\phi - 10.$$

Ex. If $b = 123$, $c = 321$, $A = 29°\,16'$, find B, C and a, having given,

$\log 99 = 1{\cdot}9956352$, $L\sin 29°\,16' = 9{\cdot}6891978$,

$\log 123 = 2{\cdot}0899051$,

$L\sin 15°\,42' = 9{\cdot}4323285$, diff. for $1' = 4492$,

$\log 222 = 2{\cdot}3463530$, $L\cot 14°\,38' = 10{\cdot}5831901$,

$\log 221 = 2{\cdot}3443923$,

$L\tan 59°\,39' = 10{\cdot}2324552$, diff. for $1' = 2898$,

we have

$$L\tan\frac{C-B}{2} = \log\tfrac{99}{222} + L\cot 14°\,38'$$

$$= 1{\cdot}9956352 - 2{\cdot}3463530 + 10{\cdot}5831901$$

$$= 10{\cdot}2324723,$$

$$\therefore \frac{C-B}{2} = 59°\,39' + \frac{10{\cdot}2324723 - 10{\cdot}2324552}{{\cdot}0002898} \times 60'',$$

or $\qquad \dfrac{C-B}{2} = 59°\,39'\,3{\cdot}5'',$

and $\qquad \dfrac{C+B}{2} = 75°\,22',$

therefore $$C = 135^\circ\ 1'\ 3{\cdot}5'',$$
$$B = 15^\circ\ 42'\ 56{\cdot}5''.$$

Again

$$\log b = \log 123 + L\sin B - L\sin A$$
$$= 2{\cdot}0899051 + 9{\cdot}6891978 - L\sin 15^\circ\ 42'\ 56{\cdot}5''.$$

And $$L\sin 15^\circ\ 42'\ 56{\cdot}5'' = L\sin 15^\circ\ 42' + \frac{56{\cdot}5}{60} \times {\cdot}0004492$$
$$= 9{\cdot}4323285 + {\cdot}0004230,$$

hence $$\log b = 2{\cdot}3463514,$$
$$b = 221 + \tfrac{19591}{19607} = 221{\cdot}992.$$

EXAMPLES. XXXII.

1. If $a = 21$, $b = 10\frac{1}{2}$, $C = 36^\circ\ 52'\ 12''$, find A and B, given

$$L\cot 18^\circ\ 26'\ 6'' = 10{\cdot}4771213,$$
$$\log 3 = {\cdot}4771213.$$

2. $a = 1$, $b = 9$, $C = 65^\circ$, find A and B.

$L\cot 32^\circ\ 30' = 10{\cdot}1958127$, $\log 2 = {\cdot}3010300$,
$L\tan 51^\circ\ 28' = 10{\cdot}0988763$,
$L\tan 51^\circ\ 29' = 10{\cdot}0991355$.

3. $a = 14$, $b = 11$, $C = 60^\circ$ given

$L\tan 11^\circ\ 44' = 9{\cdot}3174299$, $\log 4{\cdot}32 = {\cdot}6354837$,
$L\tan 11^\circ\ 45' = 9{\cdot}3180640$.

4. $b = 4$, $c = 2$, $A = 60^\circ$, find B, C and a.

$$L\tan 30^\circ = 9{\cdot}7614394,$$
$$L\cot 30^\circ = 10{\cdot}2385606.$$

5. $a=5$, $b=1$, $C=80°$.

$L \tan 38° 28' = 9{\cdot}9000865$, diff. for $1' = 2594$,

$L \cot 40° = 10{\cdot}0761865$,

$\cos 80° = {\cdot}1736482$.

6. $b=13$, $c=11$, $A=25°$, find B, C and a.

$L \cot 12° 30' = 10{\cdot}6542448$,

$L \tan 20° 36' = 9{\cdot}5750438$, diff. for $1' = 3834$.

$\log 11 = 1{\cdot}0413927$, $\log 5{\cdot}54941 = {\cdot}7442468$,

$L \sin 25° = 9{\cdot}6259483$,

$L \sin 56° 53' = 9{\cdot}9230158$, diff. for $1' = 824$.

7. $b=117$, $c=89$, $A=16°$.

$L \tan 44° 2' = 9{\cdot}9853428$, diff. for $1' = 2528$,

$L \cot 8° = 10{\cdot}8521975$, $\log 103 = 2{\cdot}0128372$,

$L \sin 16° = 9{\cdot}4403381$, $\log 89 = 1{\cdot}9493900$

$L \sin 37° 57' = 9{\cdot}7888565$, diff. for $1' = 1619$,

$\log 39{\cdot}885 = 1{\cdot}6008096$.

8. $b=8$, $c=3$, $A=62°$.

$L \tan 37° 6' = 9{\cdot}8786907$, diff. for $1' = 2626$,

$L \cot 31° = 10{\cdot}2212263$, $\log 11 = 1{\cdot}0413927$,

$L \sin 21° 53' = 9{\cdot}5713802$, diff. for $1' = 3144$,

$L \sin 62° = 9{\cdot}9459349$,

$\log 7{\cdot}1039 = {\cdot}8514968$.

9. $b=21$, $c=13$, $A=40°$.

$L \tan 32° 52' = 9{\cdot}8103025$, diff. for $1' = 2771$,

$L \cot 20° = 10{\cdot}4389341$, $\log 17 = 1{\cdot}2304489$,

$L \sin 40° = 9{\cdot}8080675$,

$L \cos 12° 52' = 9{\cdot}9889560$, diff. for $1' = 289$.

$\log 21 = 1{\cdot}3222193$,

$\log 13847 = 4{\cdot}141356$.

106. Case IV.

Having given two sides and the angle opposite one of them as a, c, A.

This is usually known as the ambiguous case.

From the equation $\sin C = \frac{c}{a} \sin A$, we find $\sin C$; when $\sin C$ is thus determined there are, in general, two values of C less than 180°, one acute and the other obtuse, whose sine has this value; *provided that* $\frac{c}{a}$ *sin* A *is* $<$ *unity.*

We must consider three different cases:

(1) if $c \sin A$ is $> a$, we have $\sin C > 1$, which is impossible, and indicates that there is no triangle with the given parts.

(2) if $c \sin A = a$, then $\sin C = 1$, and the only value of C is 90°, thus there is one triangle with the given parts, and that is a right-angled triangle.

(3) if $c \sin A < a$, then $\sin C < 1$, and there are two values of C, one acute, the other obtuse;

(α) if $c < a$, we must have $C < A$, hence C must be *acute*, and there is only one triangle with the given parts;

(β) if $c > a$, the angle C need not be acute, and both values of C may be taken, in this case then there are *two* triangles with the given parts.

We may state the above results thus:

$$\left.\begin{array}{lll} c \sin A > a & & \text{no solution} \\ c \sin A = a & & \text{one solution} \\ c \sin A < a & \left\{\begin{array}{l} c < a \\ c > a \end{array}\right. & \begin{array}{l} \text{one solution} \\ \text{two solutions} \end{array} \end{array}\right\}.$$

If $c = a$, then $C = A$ or $180° - A$; but for the latter value

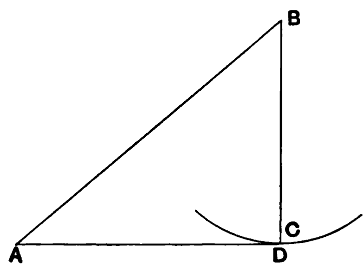
of C two sides of the triangle are coincident, the first gives the only value of C.

107. It is instructive to consider geometrically the different cases of the last article.

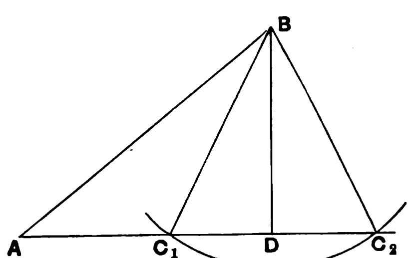
From B draw BD perpendicular to the side b, then $BD = c \sin A$; with centre B and radius a describe a circle; then

(1) if $a < c \sin A$ this circle will not cut the side AC, and there is no triangle.

(2) if $a = c \sin A$, this circle touches AC at D, and we have the right-angled triangle ADB.

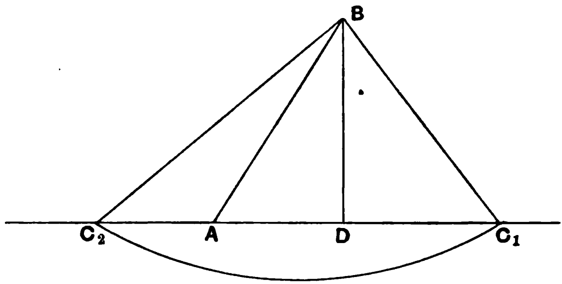


(3) if $a > c \sin A$, this circle cuts AC in two points C_1 and C_2;



(β) if $c > a$, then C_1 and C_2 lie on the same side of A and there are two triangles ABC_1 and ABC_2 having the given parts;

(a) if $c < a$, C_1 and C_2 lie on opposite sides of A and only the triangle ABC_1 has the given parts. The triangle ABC_2 has the angle at A equal to $180° - A$, and thus does not satisfy the given conditions.



108. Notice that in the second figure, of the last Article, since

$$AD = c \cos A, \text{ and } C_1D = C_2D = \sqrt{a^2 - c^2 \sin^2 A},$$

the two values of b are

$$c \cos A + \sqrt{a^2 - c^2 \sin^2 A} \text{ and } c \cos A - \sqrt{a^2 - c^2 \sin^2 A},$$

these values being both positive when there are two solutions.

We may also obtain them by solving the quadratic equation

$$a^2 = b^2 - 2bc \cos A + c^2;$$

and if we denote them by b_1 and b_2, we have

$$b_1 + b_2 = 2c \cos A,$$

$$b_1 b_2 = c^2 - a^2.$$

109. Solution of right-angled triangles.

To solve right-angled triangles we have to use logarithms in connexion with the equations already given in Chapter IV. Taking C as the right angle, we have

I. Having given a, b.

$$L\tan A = 10 + \log a - \log b,$$

$$\log c = \log a - L\sin A + 10, \quad B = 90^\circ - A.$$

II. Having given c, a.

$$L\sin A = 10 + \log a - \log c,$$

$$\log b = \tfrac{1}{2}\log(c+a) + \tfrac{1}{2}\log(c-a), \quad B = 90^\circ - A.$$

III. Having given c, A.

$$\log a = \log c + L\sin A - 10,$$

$$\log b = \log c + L\cos A - 10.$$

IV. Having given a, A.

$$\log c = \log a - L\sin A + 10,$$

$$\log b = \log c + L\cos A - 10.$$

110. Solution of triangles with other data.

We add a few examples of the solution of triangles when instead of sides and angles there are other data.

(1) Having given a, $b + c$, A.

We have

$$\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A} = \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\sin A}$$

$$= \frac{2\cos\frac{A}{2}\cos\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}.$$

Thus $\cos \frac{B-C}{2}$ is given in terms of known quantities; and therefore all the angles are known, since

$$\frac{B+C}{2} = 90^\circ - \frac{A}{2} = \text{given quantity.}$$

The problem is reduced to Case I.

(2) If the three perpendiculars from the vertices p_1, p_2, and p_3 are given.

We have, since $ap_1 = bp_2 = cp_3 = 2S$,

$$s = S\left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}\right) = S\frac{p_1p_2 + p_2p_3 + p_3p_1}{p_1p_2p_3}.$$

Similar expressions are found for $s-a$, $s-b$, and $(s-c)$.

Hence

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\frac{(p_2p_3 + p_3p_1 + p_1p_2)(-p_2p_3 + p_3p_1 + p_1p_2)}{4p_1^2p_2p_3}}.$$

In this way A is expressed in terms of the perpendiculars, and therefore determined. In like manner B and C are determined.

Ex. Solve the triangle with the following data:

(1) $B, a, b+c$. (3) p_1, B, C. (5) a, A, S.

(2) S, A, B. (4) a, bc, A.

EXAMPLES XXXIII.

1. Find A when $C = 90^\circ$, $a = 1$, $b = 2$; having given that

$$L\tan 18^\circ\, 26' = 9{\cdot}5228379,$$
$$\text{diff. for } 1' = {\cdot}0004210.$$

2. Find A and b, if $C=90°$, $c=16$, $a=5$; given that

$$L\sin 18°\ 12' = 9{\cdot}4946205, \quad \log 15{\cdot}198 = 1{\cdot}1817864,$$
$$L\sin 18°\ 13' = 9{\cdot}4950046, \quad \log 15{\cdot}199 = 1{\cdot}1818150,$$
$$\log 11 = 1{\cdot}0413927,$$
$$\log 7 = {\cdot}8450980.$$

3. In a right-angled triangle of which a side is 10 feet and the angle opposite to it 40°, find the length of the other sides.

$$L\sin 40 = 9{\cdot}8080675, \quad \log 15{\cdot}557 = 1{\cdot}1919259.$$
$$L\cos 40 = 9{\cdot}8842540, \quad \log 15{\cdot}558 = 1{\cdot}1919538,$$
$$\log 11{\cdot}917 = 1{\cdot}0761669,$$
$$\log 11{\cdot}918 = 1{\cdot}0762034.$$

4. If $a=\sqrt{2}$, $b=2$, $A=45°$, solve the triangle.

5. If $a=5$, $b=10$, $\sin A=\frac{1}{4}$, find c.

6. Given that $a=10$, $b=30$, $L\sin A = 9{\cdot}5228787$, find B.

7. Find all the angles when $b=200$, $c=100$, $A=40°$.

$$L\cot 20° = 10{\cdot}4389341, \quad L\tan 43°\ 30' = 9{\cdot}9618128.$$

8. Are the following cases ambiguous?

$$A=30°, \quad b=8, \quad a=4,$$
$$A=30°, \quad b=8, \quad a=6,$$
$$A=30°, \quad b=8, \quad a=3.$$

9. If $a=2$, $c=1+\sqrt{3}$, $A=45°$, solve the triangle.

10. If $a=11$, $b=6$, $c=7$, find C, having given that

$$L\tan 17°\ 32' = 9{\cdot}4996026, \quad \text{diff. for } 1' = 4396.$$

11. Given that $a=16$, $b=25$, $c=13$, find A.

$$L\tan 17°\ 4' = 9{\cdot}4871433, \quad \text{diff. for } 1' = 4500,$$
$$\log 11 = 1{\cdot}0413927, \quad \log 7 = {\cdot}8450980.$$

12. In a triangle

$$b = 32,\ c = 40,\ B = 52^\circ\ 32'\ 15''$$

find A and C, given that

$$L \sin 52^\circ\ 32' = 9{\cdot}8996604, \quad \text{diff. for } 1' = 968,$$
$$L \sin 82^\circ\ 50' = 9{\cdot}9965937, \quad \text{diff. for } 1' = 159.$$

13. If $a = 13$, $b = 15$, $\cos C = \frac{33}{65}$, find c, and the perpendicular upon it from the opposite angle.

14. Two ships start at the same time from the same port and sail for 5 hours at the respective rates of 8 and 10 knots an hour in straight courses inclined to each other at an angle of 60°. They then sail directly towards each other; find the inclination of this new course to their original courses, having given that

$$\log 3 = {\cdot}47712, \quad L \tan 10^\circ\ 53'\ 36'' = 9{\cdot}28432.$$

15. The sides of a triangle are 942, 812, 1270 feet. Find the area having given that

$$\log 7 = {\cdot}8450980, \quad \log 57 = 1{\cdot}7558749, \quad \log 242 = 2{\cdot}3838154,$$
$$\log 1512 = 3{\cdot}1795518, \quad \log 3{\cdot}82094 = {\cdot}58217.$$

16. Solve the triangle when

$$b = 1286, \quad c = 1093, \quad A = 6^\circ,$$

given that

$$L \tan 57^\circ\ 8' = 10{\cdot}1896975, \quad \text{diff. for } 1' = 2772,$$
$$L \cot 3^\circ = 11{\cdot}2806042,$$
$$\log 193 = 2{\cdot}2855573, \quad \log 2379 = 3{\cdot}3763944,$$
$$L \sin 29^\circ\ 51' = 9{\cdot}6969947, \quad \text{diff. for } 1' = 2201,$$
$$L \sin 6^\circ = 9{\cdot}0192346, \quad \log 22945 = 4{\cdot}3606881,$$
$$\log 1093 = 3{\cdot}0386202, \quad \log 22946 = 4{\cdot}3607070.$$

17. Find the greatest side of a triangle of which one

side is 2183 feet and whose adjacent angles are 78° 14′ and 71° 24′.

$$\log 2183 = 3{\cdot}3390537, \quad \log 42274 = 4{\cdot}6260733,$$
$$L \sin 78^\circ\, 14' = 9{\cdot}9907766, \quad \log 42275 = 4{\cdot}6260836,$$
$$L \sin 30^\circ\, 22' = 9{\cdot}7037486.$$

18. The length of one side of a triangle is 1006·62 feet and the adjacent angles are 44° and 70°. Solve the triangle, given that

$$L \sin 44^\circ = 9{\cdot}8417713, \quad \log 7654321 = 6{\cdot}8839067,$$
$$L \sin 66^\circ = 9{\cdot}9607302, \quad \log 1006{\cdot}62 = 3{\cdot}0028656,$$
$$L \sin 70^\circ = 9{\cdot}9729858, \quad \log 103543 = 5{\cdot}0151212.$$

19. Find A, B, when

$$a = 432, \quad b = 324, \quad C = 67^\circ\, 58'\, 32''.$$
$$L \cot 33^\circ\, 59' = 10{\cdot}1712851, \quad \text{diff. for } 1' = 2700,$$
$$L \tan 11^\circ\, 57' = 9{\cdot}3256073, \quad \text{diff. for } 1' = 6270.$$

20. The lengths of the sides of a triangle are 2 and 18 feet respectively, and the included angle is 38° 26′. Find the remaining angles correct to the nearest second, given that

$$L \tan 66^\circ\, 27' = 10{\cdot}3606625, \quad \text{diff. for } 1' = {\cdot}0003450,$$
$$L \cot 19^\circ\, 13' = 10{\cdot}4577187.$$

21. Find the other angles of a triangle of which one angle is 112° 4′, and the side opposite to it 573 yards long, another side being 394 yards in length.

$$L \sin 39^\circ\, 35' = 9{\cdot}8042757, \quad \log 573 = 2{\cdot}7581546,$$
$$L \sin 39^\circ\, 36' = 9{\cdot}8044284, \quad \log 394 = 2{\cdot}5954962,$$
$$L \cos 22^\circ\, 4' = 9{\cdot}9669614.$$

22. In a triangle

$$A = 20^\circ\, 41'\, 20'',$$
$$B = 51^\circ\, 38'\, 55'',$$
$$a = 24\tfrac{1}{2}.$$

Find C, having given,

$L \sin 72^\circ\, 20' = 9{\cdot}9790192$, diff. for $1' = 403$,
$L \sin 20^\circ\, 41' = 9{\cdot}5480240$, diff. for $1' = 3345$,
$\log 66078 = 4{\cdot}8200569$, $\log 245 = 2{\cdot}3891661$,
$\log 66079 = 4{\cdot}8200635$.

23. If, in the ambiguous case, we have $b_2 = 2b_1$, show that

$$3a = c\sqrt{1 + 8 \sin^2 A}.$$

24. Prove that, $\sin \dfrac{B_1 - B_2}{2} = \dfrac{b_1 - b_2}{2a}$.

25. If a, b, A be given, show that

$$\frac{\sin C}{c} = \frac{\sin C'}{c'},$$

$$\frac{\sin C}{\sin B} + \frac{\sin C'}{\sin B'} = 2 \cos A.$$

26. Show that,

$$b_1^2 - 2b_1 b_2 \cos 2A + b_2^2 = 4a^2 \cos^2 A.$$

27. If the area of one triangle be n times the area of the other,

$$\frac{a}{c} = \frac{1}{n+1} \sqrt{n^2 + 1 - 2n \cos 2A}.$$

28. If one angle of one triangle be twice the corresponding angle of the other triangle, show that

$$a\sqrt{3} = 2c \sin A, \quad \text{or} \quad 4c^3 \sin^2 A = a^2 (a + 3c).$$

CHAPTER XVI.

HEIGHTS AND DISTANCES.

111. IN Chapter IV. we gave some illustrations of the application of trigonometry in simple cases to determine *heights* and *distances*. In the present chapter we give additional examples, of a somewhat less elementary nature, making use of the methods for solving triangles which have been discussed in Chapter XV.

112. To find the height of a distant point.

Let P (see next page) be the distant point, and C the foot of the perpendicular from P on the horizontal plane which contains A, where the observer is supposed to be situated.

To measure PC, which will be called h, we may proceed in either of three ways, illustrated by the figures.

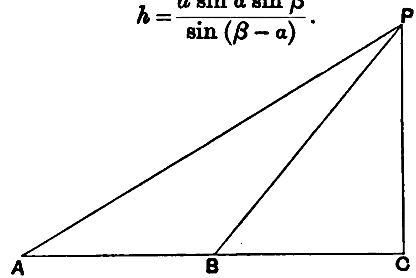
(i) Let a line $AB = a$, called a *base line*, be measured *towards* PC, and let the elevations of P at A and B be observed. Calling them α and β, we have since

$$a = AC - BC,$$

$$a = h \cot \alpha - h \cot \beta$$

$$= h\left(\frac{\cos \alpha}{\sin \alpha} - \frac{\cos \beta}{\sin \beta}\right) = \frac{h \sin(\beta - \alpha)}{\sin \alpha \sin \beta},$$

hence $$h = \frac{a \sin \alpha \sin \beta}{\sin(\beta - \alpha)}.$$

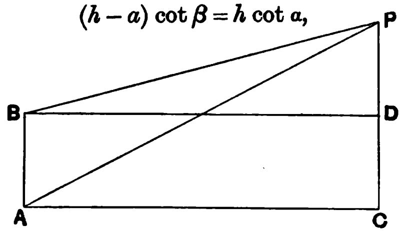


Taking logs. we get,

$$\log h = \log a + L \sin \alpha + L \sin \beta - L \sin(\beta - \alpha) - 10.$$

(ii) Let AB be measured vertically upwards. Draw BD parallel to AC. Then if α and β are the elevations of P at A and B respectively, since $BD = AC$, we have

$$(h - a) \cot \beta = h \cot \alpha,$$



or $$h = \frac{a \cot \beta}{\cot \beta - \cot \alpha} = \frac{a \cos \beta \sin \alpha}{\sin(\alpha - \beta)}.$$

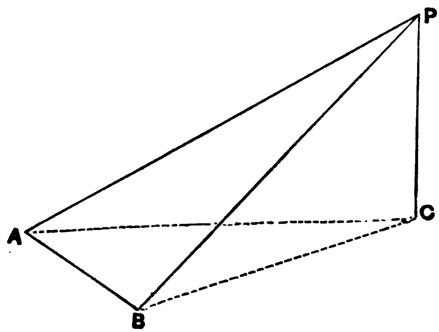
Taking logs. we get,

$$\log h = \log a + L \cos \beta + L \sin \alpha - L \sin(\alpha - \beta) - 10.$$

(iii) If it is not possible to measure the base line in either of these ways, let it be measured in any other direction: let the elevation α of P, be measured at A, and also the angles $PAB = \gamma$, and $PBA = \delta$, then since

$$\frac{PA}{\sin \delta} = \frac{AB}{\sin APB},$$

we have $PA = a\dfrac{\sin\delta}{\sin(\gamma+\delta)}$, also $h = AP\sin\alpha$,



hence $$h = a\frac{\sin\alpha\sin\delta}{\sin(\gamma+\delta)}.$$

Thus h is determined.

Ex. 1. Observations to find the height of a mountain are taken at two stations A and B which are at the same height and 4000 feet apart. The elevation of the top at A is found to be 60°; and the angles PAB and PBA are found to be 75° and 60° respectively, where P is the top. Find the height of the mountain.

Using the formula

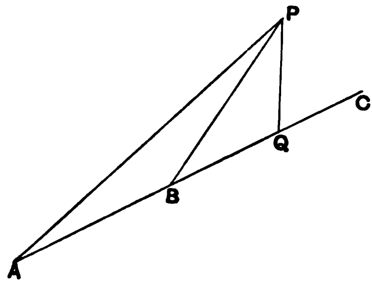
$$h = a\frac{\sin\alpha\sin\delta}{\sin(\gamma+\delta)},$$

since here $a = 4000$, $\alpha = \delta = 60°$, $\gamma+\delta = 135°$,

$$\text{height required} = 4000\,\frac{\left(\frac{\sqrt{3}}{2}\right)^2}{\cos 45°}$$

$$= 4000 \times \frac{3}{\sqrt{2}} = 6000 \times \sqrt{2} = 8486 \text{ feet nearly.}$$

Ex. 2. A tower PQ stands on a hill which is inclined to the vertical at an angle α. At two points A and B on the side of the hill, in the same vertical plane as the tower, the angles subtended by the tower are β and γ. The distance AB is found to be a. Find the height of the tower.



The angle PQC is given equal to α, and the angles PAQ, PBQ are equal to β and γ respectively. Let h be the length of PQ.

We have from the $\triangle\ PAQ$,

$$\frac{h}{\sin\beta}=\frac{AQ}{\sin(\alpha-\beta)},\quad \therefore\ \frac{h\sin(\alpha-\beta)}{\sin\beta}=AQ,$$

and from the $\triangle\ PBQ$,

$$\frac{h}{\sin\gamma}=\frac{BQ}{\sin(\alpha-\gamma)},\quad \therefore\ \frac{h\sin(\alpha-\gamma)}{\sin\gamma}=BQ;$$

$$\therefore\ AQ-BQ=a=h\left\{\frac{\sin(\alpha-\beta)}{\sin\beta}-\frac{\sin(\alpha-\gamma)}{\sin\gamma}\right\},$$

which determines h.

The following example shows the application of logarithms; in (i) let it be given that

$$a=1000\text{ yds.},\quad \alpha=30°,\quad \beta=45°,$$

then $\log h=\log 1000+L\sin 30°+L\sin 45°-L\sin 15°-10.$

By reference to Tables we find that,

$L\sin 45°=9{\cdot}84948$, $L\sin 15°=9{\cdot}41299$, $L\sin 30°=9{\cdot}69897$,

from which, $\log h=3{\cdot}13545$; and since $\log 1366=3{\cdot}13545$, we see that $h=1366$ yds.

EXAMPLES XXXIV.

1. A man observes the elevation of a tower to be 15°, he walks directly towards it for a distance c, when he finds the elevation to be 75°. Show that the height is $\frac{c}{2\sqrt{3}}$.

2. The elevations of the tower and spire of a church are 30° and 60° respectively; a feet nearer the elevation of the top of the tower is 60°, show that the heights of the tower and spire are

$$\frac{\sqrt{3}a}{2}, \quad \sqrt{3}a \text{ feet.}$$

3. At a point on a level plain a tower subtends an angle α, and a man b feet high on its top an angle ϵ, prove

$$\text{height of tower} = \frac{b \sin \alpha \cos (\alpha + \epsilon)}{\sin \epsilon}.$$

4. From the deck of a ship the angle of elevation of the top of a mountain is 41°, and from the mast-head it is 40°. If the height of the mast is 100 feet, find the height of the mountain.

$$L \cos 40° = 9{\cdot}8842540, \qquad L \sin 1° = 8{\cdot}2418553,$$
$$L \sin 41° = 9{\cdot}8169429, \qquad \log 28797 = 4{\cdot}4593472.$$

5. At a certain place the elevation of the top of a tower is 20°, and 200 feet nearer the elevation is 25°. Find the height of the tower.

$$L \sin 20° = 9{\cdot}5340517, \qquad \log 33169 = 4{\cdot}5207324,$$
$$L \sin 25° = 9{\cdot}6259483, \qquad \log 33170 = 4{\cdot}5207455,$$
$$L \sin \ 5° = 8{\cdot}9402960.$$

6. From the top of a hill the depression of the nearest point of a circular pond is 48°, and that of the furthest point is 45°. The height of the hill above the pond being 300 feet, find the radius of the pond.

$L \sin 45° = 9{\cdot}8494850$, $\log 14939 = 4{\cdot}1743215$,
$L \sin 48° = 9{\cdot}8710735$, $\log 14940 = 4{\cdot}1743506$,
$L \sin 3° = 8{\cdot}7188002$.

7. A man sees a fort 26° N. of E., and after walking 2000 yards in a direction 40° S. of E. he then sees it due N. Find the distance of the fort from his second position.

$L \sin 66° = 9{\cdot}9607302$, $\log 20328 = 4{\cdot}3080947$,
$L \sin 64° = 9{\cdot}9536602$, $\log 20329 = 4{\cdot}3081160$.

8. From the top of a cliff the depressions of two buoys in the same vertical plane, whose distance apart is known to be half a mile, are 38° and 15°; find the height of the cliff.

$L \sin 38° = 9{\cdot}78934$, $\log 2039 = 3{\cdot}30942$,
$L \sin 15° = 9{\cdot}41299$,
$L \sin 23° = 9{\cdot}59187$.

9. From two places on the line of the sea-shore 5000 yards apart, a ship is seen in directions making angles of 35° and 47° with the shore-line. Find its distance from the shore; given that

$L \sin 35° = 9{\cdot}7585913$, $\log 21180 = 4{\cdot}3259260$,
$L \sin 47° = 9{\cdot}8641275$, $\log 21181 = 4{\cdot}3259465$,
$L \sin 82° = 9{\cdot}9957528$.

10. A slender tower surmounted by a flagstaff stands on a level plain. From a certain point in the plain the tower is seen to subtend an angle β, and the flagstaff an angle α. From a second point a feet nearer the base the flagstaff again subtends an angle α.

Show that the height of the tower is

$$\frac{a \tan \beta}{1 - \tan \beta \tan (\alpha + \beta)}.$$

11. A man standing on the bank of a river observes the elevation of the top of a tree on the opposite bank to be 51°, and when he retires 30 feet from the river's edge he finds the elevation to be 46°, find the breadth of the river having given

$$\log 1{\cdot}558 = {\cdot}19263, \qquad L \sin 46^\circ = 9{\cdot}85693,$$
$$\log 3 = {\cdot}47712, \qquad L \sin 39^\circ = 9{\cdot}79887,$$
$$L \sin 5^\circ = 8{\cdot}94029.$$

12. $ABCD$ is a rectangular piece of water whose area is required; but the only measures that can be taken are the angles which BC subtends at A and at a point P which is 220 feet from A in BA produced, the former angle being 71° and the latter 55°. Find the length and breadth of the rectangle having given,

$$\log 2{\cdot}2 = {\cdot}34242, \qquad L \sin 16^\circ = 9{\cdot}44034,$$
$$\log 2{\cdot}1285 = {\cdot}32808, \qquad L \sin 55^\circ = 9{\cdot}91336,$$
$$\log 6{\cdot}1817 = {\cdot}79111, \qquad L \sin 71^\circ = 9{\cdot}97567,$$
$$L \cos 71^\circ = 9{\cdot}51264.$$

13. There are two places A and B on opposite sides of a vertical tower situated on a hill; the distance of A from the foot of the tower being a feet, that of B being b feet. The

angles subtended by the tower at A and B are a and β respectively. If h is the height of the tower show that

$$a + b = h(\cot a + \cot \beta)\cos \epsilon,$$

if ϵ be the slope of the hill to the horizon, and that

$$2\sin\epsilon = (\cot a - \cot\beta)\cos\epsilon + \frac{b-a}{h}.$$

14. AB is a horizontal line whose length is 400 yds.; from a point in the line between A and B a balloon ascends vertically, and after a certain time its altitude is taken simultaneously from A and B; at A it is observed to be $64° 15'$, at B $48° 20'$; find the height of the balloon when the observation was taken, given that

$$L\sin 64° 15' = 9{\cdot}9545793, \quad L\sin 48° 20' = 9{\cdot}8733352,$$
$$L\sin 67° 25' = 9{\cdot}9653532, \quad \log 29149 = 4{\cdot}4646213,$$
$$\log 2 = {\cdot}3010300.$$

15. An observer is situated in a boat vertically beneath the centre of a suspension bridge. He finds that its length subtends at his eye an angle $2a$. At another point distant a down stream which is immediately opposite the centre of the bridge, he finds it subtends an angle 2β. Show that the length of the bridge is

$$\frac{a}{\sqrt{\cot^2\beta - \cot^2 a}}.$$

16. A man observes that when he has walked c feet up an inclined plane the depression of an object beneath him, in the horizontal plane through the foot of the inclined plane is a; after he has walked a further distance of c feet he finds its depression to be β.

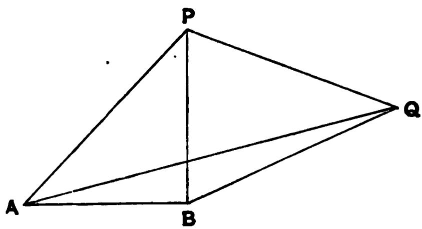
Show that the inclination of the slope to the horizon is θ, where

$$\cot\theta = 2\cot\beta - \cot a.$$

17. A coast-guardsman standing on a cliff observes a ship due west of him whose depression is α. An hour later the ship is due south and its depression is β. If h feet is the height of the cliff show that the ship is moving at the rate of $h\sqrt{\cot^2\alpha + \cot^2\beta}$ feet per hour.

113. To find the distance between two inaccessible points.

Let P and Q be the two objects, and let any base line $AB = a$ be measured, the points A and B being so chosen that P and Q are both visible from each of them.



At A measure the three angles $PAQ = \alpha$, $QAB = \beta$, $PAB = \gamma$; it should be observed that the triangles PAQ, QAB are not in general in the same plane, if they are we have $\gamma = \alpha + \beta$.

At B measure the angles $PBA = \delta$ and $QBA = \epsilon$.

From the two triangles ABP, ABQ, we have

$$AP = a\frac{\sin\delta}{\sin(\gamma+\delta)},$$

$$AQ = a\frac{\sin\epsilon}{\sin(\beta+\epsilon)}.$$

Thus AP, AQ are determined by the formulae

$$\log AP = \log a + L\sin\delta - L\sin(\gamma+\delta),$$

$$\log AQ = \log a + L\sin\epsilon - L\sin(\beta+\epsilon).$$

In the triangle PAQ we now know AP, AQ and the angle PAQ, or α.

We therefore find the angles APQ, AQP, as in Art. 105, thus

$$L\tan\frac{APQ-AQP}{2}=L\cot\frac{\alpha}{2}+\log(AQ-AP)-\log(AQ+AP),$$

$$APQ+AQP=180^\circ-\alpha.$$

And PQ is then found by use of the formula

$$\log PQ=\log AP+L\sin\alpha-L\sin AQP.$$

EXAMPLES. XXXV.

1. A ship S is observed simultaneously by two observers at two points A and B on the line ABC of the shore, distant one mile from each other. At a given time the angles CAS, CBS are observed to be $27^\circ\,8'$ and $32^\circ\,22'$ respectively. Half an hour afterwards the corresponding angles are $55^\circ\,35'\,51''$ and $62^\circ\,22'$. Show that the distance traversed by the ship in the interval is 3·7 miles nearly, having given

$$L\sin 55^\circ\,35'\,51''=9{\cdot}9165007,$$
$$L\sin 6^\circ\,46'\,9''=9{\cdot}0714016,$$
$$L\sin 27^\circ\,8'=9{\cdot}6590246,$$
$$L\sin 5^\circ\,14'=8{\cdot}9600517,$$
$$\log 5={\cdot}6989700,\quad \log 7={\cdot}8450980.$$

2. A, B, C, D are four points in the same horizontal plane. At A the angles α and β are subtended by CD and CB, at B the angles α and γ are subtended by CD and DA, and AB is known to be a. Show that

$$CD=a\sin\alpha\div\sin(\alpha+\beta+\gamma).$$

3. In a survey it is found necessary to continue a line AB beyond an obstacle at B. A line BD is measured at right angles to AB, and from D lines DP, DQ are drawn which clear the obstacle. The angles BDP, BDQ are measured and are found to contain 41° and 68° respectively, and the distance BD is 180 yards.

What lengths must be set off along DP and DQ to ensure that PQ will lie in the prolongation of AB.

$$\log 1{\cdot}8 = {\cdot}255273, \quad L\cos 41^\circ = 9{\cdot}877780,$$
$$\log 2{\cdot}385 = {\cdot}377488, \quad L\sin 22^\circ = 9{\cdot}573575,$$
$$\log 4{\cdot}805 = {\cdot}681693.$$

4. A person walking along a straight road watches two spires till they appear in the same straight line, and finds that this line makes an angle β with the road. From the point where this is the case he walks a yards, when the nearest spire lies in a direction perpendicular to the road, and at this point he observes that the angle subtended by the two spires is α. Show that the distance in yards between the two churches is

$$a\left\{\frac{\cos\alpha}{\cos(\alpha+\beta)} - \frac{1}{\cos\beta}\right\}.$$

5. The angular elevation of a column viewed from a station due north of it being α, and from a station due east of the former station and at a distance c from it being β, show that the height of the tower is

$$\frac{c\sin\alpha\sin\beta}{\sqrt{\sin(\alpha+\beta)\sin(\alpha-\beta)}}.$$

6. At one end of a base line one mile in length the elevation of the top of a mountain is observed to be 5° 25′,

and the angle subtended by its summit and the other end of the base line to be $93° \; 22'$.

At the other end of the base line the angle subtended by the top and the first station is $75°$. Find the distance of the first station from the top and show that $6{\cdot}782718 \times 352$ is the vertical height of the mountain in feet, having given that

$$\begin{aligned} L\sin 75° &= 9{\cdot}984944, & \log 15 &= 1{\cdot}176091, \\ L\sin 11°\,38' &= 9{\cdot}304593, & \log 67{\cdot}83 &= 1{\cdot}831422, \\ L\cos\ 5°\,25' &= 9{\cdot}998056, & \log 67{\cdot}82 &= 1{\cdot}831358, \\ L\sin\ 5°\,25' &= 8{\cdot}974962, & \log 71{\cdot}54 &= 1{\cdot}854549, \\ & & \log 71{\cdot}53 &= 1{\cdot}854488. \end{aligned}$$

7. At each end of a base line of length $2a$ it is found that the angular altitude of a certain peak is θ, and at the middle point of the base the altitude is ϕ. Prove that the vertical height of the peak above the plane is

$$\frac{a \sin\theta \sin\phi}{\sqrt{\sin(\theta+\phi)\sin(\theta-\phi)}}.$$

8. A light-house facing N. sends out a fan-shaped beam extending from N.E. to N.W. A steamer sailing due W. first sees the light when 5 miles away from the light-house and continues to see it for $30\sqrt{2}$ minutes. What is its speed?

9. On the top of a chimney of height h the angles of depression of two objects A and B on the horizontal plane are $\frac{\pi}{4}-a$ and $\frac{\pi}{4}+a$, while the angle subtended by AB is $2a$. [A and B are not in a line with the foot of the tower.]

Show that $\qquad 2h \tan 2a = AB.$

10. A flagstaff stands in the middle of a square tower. A man on the ground opposite the middle of a side of the tower, and distant 100 feet from it just sees the flag;

receding another 100 feet the elevations of the tops of the tower and flag are α and β respectively, where

$$\tan\alpha = \tfrac{1}{2}, \quad \tan\beta = \tfrac{5}{9}.$$

Find the heights of the tower and flagstaff.

11. The elevation of a tower from a point A due N. of it is observed to be 45°, and from a point B due E. to be 30°. If $AB = 240$ feet, find the height of the tower.

12. If from a point at the foot of the mountain, at which the elevation of the observatory on Ben Nevis is 60°, a man walks 1900 feet up a slope of 30°, and then finds that the elevation of the observatory is 75°, show that the height of Ben Nevis is nearly 4500 feet.

13. A man stands on the top of a wall of height h feet and observes the elevation of a telegraph post to be α, he then descends from the wall and finds the elevation to be β, show that the height of the post exceeds that of the man by $\dfrac{h\sin\beta\cos\alpha}{\sin(\beta-\alpha)}$ feet.

14. A tower stands at the foot of a hill whose inclination to the horizon is 9°. From a point 100 feet up the hill the tower is seen to subtend an angle of 54°. Find the height of the tower.

15. A light-house is seen N. 20° E. from a vessel sailing S. 25° E. and a mile further on it appears due N. Determine its distance at the last observation, having given,

$$L\sin 20° = 9{\cdot}534052, \qquad \log 2 = {\cdot}3010300,$$
$$\log 206 = 2{\cdot}313867, \quad \log 207 = 2{\cdot}315900.$$

16. From the top of a hill the depression of a point in the plain below is 30°, and from a spot three-quarters of

the way down the depression of the same point is 15°, if α is the inclination of the hill show that $\tan\alpha = \dfrac{3}{3\sqrt{3}-2}$.

17. A tower is situated on a horizontal plane at a distance a from the base of a hill whose inclination is α. A person on the hill, looking over the tower, can just see a pond, the distance of which from the tower is b. Show that, if the distance of the observer from the foot of the hill be c, the height of the tower is $\dfrac{bc\sin\alpha}{a+b+c\cos\alpha}$.

18. A column lying prostrate on a horizontal plane subtends a right angle at the summit of a tower standing on the plane, the angles of depression of the ends of the column being α and β, prove that if h is the height of the tower,

$$\text{length of column} = h\,(\operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta)^{\frac{1}{2}}.$$

19. Two stations A and B are in the same vertical plane with a third station C. The altitude h of A above the horizontal plane through C is known, and that of B, (H) is required.

The inclinations of AC, BC, AB to the vertical are observed to be respectively α, β, and γ. Prove that

$$H = h\,\frac{\cos\beta}{\cos\alpha}\cdot\frac{\sin(\gamma-\alpha)}{\sin(\gamma-\beta)}.$$

20. Two land-marks exactly N. and S. of one another are separated by a river. A person walks due W. from one of them a distance a, and then a further distance b, he finds that the angle subtended at his eye by the objects in the first case is three times the angle subtended in the second

case. Show that the distance between the land-marks is equal to

$$(a+b)\sqrt{\frac{b-2a}{3b+2a}}.$$

21. There are three telegraph posts A, B and C, 100 yards apart. A man at P observes the angles APB, BPC to be $44^\circ\, 42'\, 3''$ and $19^\circ\, 17'\, 57''$. Find the angles PAB and PCB, and also his distance from B, having given that

$$L\tan 32^\circ = 9{\cdot}79578, \qquad \log 3 = {\cdot}47712,$$
$$L\tan 12^\circ\, 42'\, 3'' = 9{\cdot}35301, \qquad \log 1420 = 3{\cdot}15229,$$
$$L\sin 88^\circ = 9{\cdot}99973, \qquad \log 1421 = 3{\cdot}15259,$$
$$L\sin 44^\circ\, 42'\, 3'' = 9{\cdot}84721.$$

22. On a wide plain there stand two distant pillars C and D. At two stations A and B observations are made and it is found that at A the angle subtended by BD is a right angle, while the angles DAC, ABD and CBD are each equal to α.

If $AB = a$, show that $CD = a\tan\alpha$.

23. A pole standing in a horizontal plane leans over towards the S. At equal distances due N. and S. of it, the elevations of its summit are α and β. Show that it is inclined to the vertical at an angle θ, where

$$\cot\theta = \frac{\sin(\beta-\alpha)}{2\sin\alpha\sin\beta}.$$

24. A column stands on a flat plain on which three stations A, B, and C, are taken in a straight line with the foot of the column; AC is equal to a, and BC to b. The column viewed from A subtends an angle 2θ, viewed from

B an angle $\frac{\pi}{2}-\theta$, and from C an angle θ. Show that the height of the column is $\sqrt{\left(a+\frac{b}{2}\right)\left(a-\frac{b}{2}\right)}$.

25. A person in a ship observes that the line joining two buoys at sea subtends an angle α. The ship then sails a distance a in a direction at right angles to the line joining the buoys and it is found that the angle subtended is the same as before. When she has sailed a further distance c she is in a line with the buoys, prove that the distance between the buoys is $(a+2c)\tan\alpha$.

26. A tower on the side of a hill is observed to subtend the same angle at two points on the same side of it on a horizontal plane. Show that if the elevations of the top of the tower at the two places are α and β, its height is

$$\frac{a\cos(\alpha+\beta)}{\sin(\alpha-\beta)},$$

where a is the distance between the points, which are in the same vertical plane as the tower.

27. From each of two points A and B on the same horizontal plane the top of a tower C and the peak D of a mountain are observed to subtend the same angle α, and from the point C, AB subtends the angle γ. Also β is the altitude of the tower, from a point in the horizontal plane in which A and B lie, at which the top of the mountain is seen just behind C.

Show that the height of the mountain is greater than that of C by $AB\dfrac{\sin\alpha\sin\beta}{\sin\gamma}$.

28. A building on a square base $ABCD$ has two of its sides AB and CD parallel to the bank of a river. An observer standing on the opposite bank in the same straight

line with DA, finds that AB subtends at his eye an angle of 45°. Having walked a yards along the bank he finds that the side DA subtends an angle whose sine is $\frac{1}{3}$.

Show that the length of each side is $\dfrac{a}{\sqrt{2}}$ yards.

29. A man ascends the slope of a hill inclined at an angle α to the horizon, and twice during the ascent at vertical heights a and b above the level of the plain he turns round and observes the angle of depression of the top of a tower standing on the plain directly in front of him. If β and γ are these angles, show that the height of the tower is

$$\{b(\cot\alpha - \cot\gamma) - a(\cot\alpha - \cot\beta)\} \div (\cot\beta - \cot\gamma).$$

30. The angular elevation of a tower at a place A due south of it is 30°, and at a place B due west of A, and at a distance a from it, the elevation is 18°, show that the height of the tower is $\dfrac{a}{\sqrt{2\sqrt{5}+2}}$.

31. Two poles a and b feet long respectively are placed vertically in a horizontal plane so as to subtend the angle α at a point in the line joining their feet.

If β and β' be the angles which they subtend at any point in the horizontal plane at which the line joining their feet subtends a right angle, show that

$$(a+b)^2 \cot^2\alpha = a^2 \cot^2\beta + b^2 \cot^2\beta'.$$

32. The angular elevations of a balloon were observed simultaneously from three stations A, B and C in the same straight line, and found to be α, β and γ. If $AB = a$, $BC = c$, $AC = b$, show that the height of the balloon is

$$\sqrt{\frac{abc}{a\cot^2\gamma + c\cot^2\alpha - b\cot^2\beta}}.$$

33. Three towers of heights a, b and c stand in the same straight line at distances d and e, prove that they cannot subtend equal angles at any point in the plane unless

$$(a^2 - b^2)\,e \text{ is } > (b^2 - c^2)\,d.$$

34. The plane side of a hill which runs E. and W. is inclined to the horizon at an angle α. It is required to construct a straight railway on it inclined at an angle β to the horizon. Show that the point of the compass towards which it will be directed is θ° W. of N., where

$$\cos\theta = \cot\alpha \,.\, \tan\beta.$$

35. A flagstaff on a tower is observed from two points in the same horizontal plane, one due S. and the other due E. of it, and is seen to subtend the same angle at each point. The angles of elevation of the top of the flagstaff are measured at the same time and are found to be α and β, where $\tan\alpha = a$, $\tan\beta = b$. If c be the distance between the points of observation, show that the height of the flagstaff $= c \,.\, \dfrac{ab-1}{\sqrt{a^2+b^2}}$.

36. An inaccessible tower stands in the centre of a circular enclosure, and its angle of elevation as observed from a point A on the circumference of the circle is m°. Also the angle between the two lines drawn from A to the base of the tower and to a point P on the circumference is n°. The arc AP is measured and found to be a yards. Show that the height of the tower in yards is $a \tan m \div 2\theta$, where θ is the circular measure of the complement of n°.

37. From a house on one side of a road observations are made of the angles subtended by the houses opposite, first from the level of the road and next from a room

window at a height c. If these angles are α and β show that h, the height of the house, is given by the equation

$$\frac{c^2}{h^2} - \frac{c}{h} = \cot\beta\cot\alpha - \cot^2\alpha.$$

38. A man walking along a straight road which runs in a direction 30° to the E. of N. notes when he is due S. of a certain house. When he has walked a mile further, he observes that the house lies due W., and that a windmill on the opposite side of the road is N.E. of him. Three miles further on he finds that he is due N. of the windmill. Find the distance between the house and the windmill, and show that the line joining them makes with the road an angle whose tangent is

$$\frac{48 - 25\sqrt{3}}{11}.$$

39. The sky being covered with a stratum of cloud one mile above the earth, if the altitude of a point on the edge of a cloud be α, show that its distance from the eye, in miles, is nearly

$$\frac{1}{\sin\alpha}\left(1 - \frac{\cot^2\alpha}{2N}\right),$$

where N = number of miles in the earth's radius.

40. Using the notation of Art. 113, show that the area of the quadrilateral $ABQP$ is

$$\frac{a^2 \sin\gamma \sin\epsilon \sin(\beta + \delta)}{2\sin(\beta + \epsilon)\sin(\gamma + \delta)};$$

the points A, B, P, Q, being supposed all in the same plane.

41. In the middle of a field which has the shape of an equilateral triangle there stands a tower 200 feet high. From the top of the tower each side of the field subtends

an angle whose cosine is $\frac{1}{10}$. Show that the length of a side of the field is nearly 424 feet.

42. A statue on the top of a pillar situated on level ground is observed to subtend the greatest angle α at the eye of an observer when his distance from the pillar is c feet; prove that the length of the statue is $2c \tan \alpha$ feet.

43. A large cylindrical column lies on the ground. An observer notices the angle subtended by the section of the column at the level of the ground to be α. On approaching a feet nearer, the angle subtended is found to be β. Show that the radius of the column is

$$\frac{a \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\sin \frac{\beta - \alpha}{2}}.$$

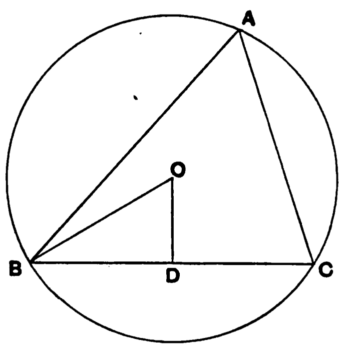
44. From a certain station the elevation of a mountain peak in the N. E. is observed to be α. A hill in the E. S. E. whose height above the station is known to be h, is then ascended, and the mountain peak is now seen in the N. at an elevation β. Prove that the height of its summit above the first station is $h \sin \alpha \cos \beta \operatorname{cosec} (\alpha - \beta)$.

CHAPTER XVII.

PROPERTIES OF TRIANGLES.

114. The circumscribed circle of a triangle.

To find the radius of the circle circumscribing a triangle, or, as it is called, the *circum-circle*, we proceed as follows :—



Let O be the *circum-centre;* draw OD perpendicular to the side BC of the triangle ABC, then D is the middle point of BC, and $\angle BOD = A$. Let R be OB, or OC.

Since $BD = OB \sin BOD$, we have

$$\tfrac{1}{2}a = R \sin A, \quad \text{or} \quad R = \frac{a}{2 \sin A} \quad \text{.........(1).}$$

Also since $$\frac{a}{2\sin A} = \frac{abc}{2bc\sin A} = \frac{abc}{4S},$$

where S is the area of ABC, Art. 89.

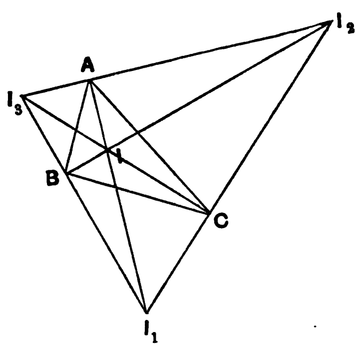
We have $$R = \frac{abc}{4S} \quad\text{.......................(2).}$$

Notice that $$OD = OB\cos A = R\cos A.$$

115. The inscribed and escribed circles.

Four circles can be described so as to touch the three sides of a triangle; the inscribed circle, or *in-circle*, touches each side internally. Its centre I is the point of intersection of the lines bisecting the *internal* angles of the triangle. Euc. IV. 4.

The three points I_1, I_2, I_3, where the external bisectors intersect, are the centres of circles each of which touches one side of the triangle and the other two sides produced.



We know that IA, IB, and IC being produced pass through the points I_1, I_2 and I_3 respectively. Hence we observe that

(i) AI_1, BI_2, CI_3 are the perpendiculars from the vertices of the triangle $I_1I_2I_3$ on the opposite sides.

(ii) I is the orthocentre (or intersection of perpendiculars), of the triangle $I_1I_2I_3$.

(iii) The circum-circle of ABC is the nine-point circle of $I_1I_2I_3$.

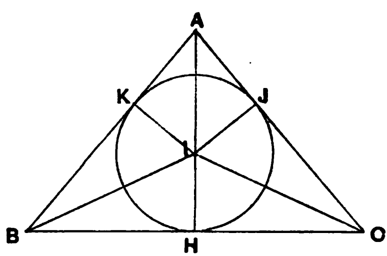
Hence this circle passes through the middle points of its sides and also through the middle points of II_1, II_2, and II_3.

116. Radii of the inscribed and escribed circles.

If r is the radius of the in-circle, and r_1 that of the escribed circle which touches the side a; we shall show that

$$r = \frac{S}{s}; \qquad r_1 = \frac{S}{s-a}.$$

Let H, J, K be the points of contact of the in-circle of the triangle ABC.



Then $\qquad \triangle IBC + \triangle ICA + \triangle IAB = S.$

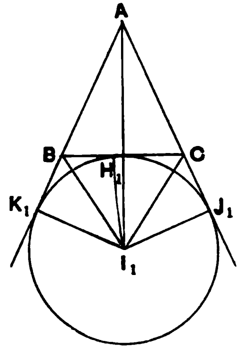
Now $\qquad \triangle IBC = \frac{1}{2} IH \,.\, BC = \frac{1}{2} r \,.\, a,$

$$\triangle ICA = \frac{1}{2} IJ \,.\, CA = \frac{1}{2} r \,.\, b,$$

$$\triangle IAB = \frac{1}{2} IK \,.\, AB = \frac{1}{2} r \,.\, c;$$

$$\therefore \; S = \frac{1}{2} r (a+b+c) = r \,.\, s \quad \text{...........(3)}.$$

Again let H_1, J_1, K_1 be the points of contact of the escribed circle whose centre is I_1.



Then $\quad \triangle I_1AC + \triangle I_1AB - \triangle I_1BC = S.$

Also $\quad \triangle I_1AC = \frac{1}{2} I_1J_1 \,.\, AC = \frac{1}{2} r_1 \,.\, b,$

$$\triangle I_1BC = \tfrac{1}{2} I_1K_1 \,.\, AB = \tfrac{1}{2} r_1 c,$$

$$\triangle I_1BC = \tfrac{1}{2} I_1H_1 \,.\, BC = \tfrac{1}{2} r_1 a,$$

$$\therefore \; S = \tfrac{1}{2} r_1 (b + c - a) = r_1 (s - a) \quad \text{.........(4).}$$

In the same way we find that

$$S = r_2 (s - b) = r_3 (s - c).$$

117. The lengths of the lines intercepted between the vertices and the points of contact of these circles are

$$s, \; s - a, \; s - b, \; s - c.$$

For in the first figure of the last Article, since

$$AK + BC = \tfrac{1}{2} (BC + CA + AB) = s,$$

$$\therefore \; AK = AJ = s - a.$$

Similarly $\quad BH = BK = s - b,$

$$CJ = CH = s - c.$$

And in the other figure, since

$$AK_1 + AJ_1 = AB + BC + CA = 2s,$$

$$\therefore\ AK_1 = AJ_1 = s.$$

Also $$BK_1 + AB = AK_1 = s,$$

$$\therefore\ BK_1 = BH_1 = s - c.$$

Similarly $$CJ_1 = CH_1 = s - b.$$

118. Other expressions for the radii of the inscribed and escribed circles may be obtained. Thus, referring to the first figure of Art. 116,

$$a = BH + HC = r\,(\cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C)$$

$$= r\,\frac{\sin \dfrac{B+C}{2}}{\sin \dfrac{B}{2} \sin \dfrac{C}{2}} = r\,\frac{\cos \dfrac{A}{2}}{\sin \dfrac{B}{2} \sin \dfrac{C}{2}},$$

whence $$r = a \sin \frac{B}{2} \sin \frac{C}{2} \sec \frac{A}{2} \quad \text{..............(5)}.$$

Combining the formulae (1) and (5) we obtain

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \text{..............(6)}.$$

Also since r equals each of the expressions

$$AJ \tan \frac{A}{2}, \quad BK \tan \frac{B}{2}, \quad CH \tan \frac{C}{2},$$

we have

$$r = (s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2} \text{...(7)}.$$

Similarly from the second figure

$$a = BH_1 + CH_1 = r_1\,(\tan \tfrac{1}{2}B + \tan \tfrac{1}{2}C)$$

$$= r_1\,\frac{\sin \dfrac{B+C}{2}}{\cos \dfrac{B}{2} \cos \dfrac{C}{2}} = r_1\,\frac{\cos \dfrac{A}{2}}{\cos \dfrac{B}{2} \cos \dfrac{C}{2}},$$

whence $$r_1 = a \cos\frac{B}{2}\cos\frac{C}{2}\sec\frac{A}{2} \quad\text{..............(8).}$$

Combining (1) and (8) we have

$$r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \quad\text{...........(9).}$$

And since r_1 equals each of the expressions

$$AK_1\tan\frac{A}{2},\quad BH_1\cot\frac{B}{2},\quad CJ_1\cot\frac{C}{2},$$

we have

$$r_1 = s\tan\frac{A}{2} = (s-c)\cot\frac{B}{2} = (s-b)\cot\frac{C}{2} \quad\text{...(10).}$$

EXAMPLES XXXVI.

1. Find the radii of the inscribed and escribed circles of the triangles whose sides are

(i) 4, 6, 8. (ii) 7, 10, 15. (iii) 5, 7, 10. (iv) 4, $6\frac{1}{2}$, $3\frac{1}{2}$. (v) 5, 6, 9.

In any triangle ABC prove the following:

2. $r_1 + r_2 + r_3 - r = 4R.$

3. $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}.$

4. $r_2r_3 + r_3r_1 + r_1r_2 = \dfrac{S^2}{r^2}.$

5. $r_1 = r\cot\dfrac{B}{2}\cot\dfrac{C}{2}.$

6. $rr_1 = r_2r_3\tan^2\dfrac{A}{2}.$

7. Prove that the following expressions are each equal to S.

(i) $\sqrt{rr_1r_2r_3}$. (ii) $rr_1\dfrac{r_2-r_3}{b-c}$.

(iii) $r_1r_2\sqrt{\dfrac{4R-(r_1+r_2)}{r_1+r_2}}$. (iv) $r_2r_3\tan\dfrac{A}{2}$.

(v) $r^2\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$.

8. Prove the following expressions for the sides and angles of a triangle in terms of the radii of the escribed circles.

(α) $a=\dfrac{r_1(r_2+r_3)}{\sqrt{r_2r_3+r_3r_1+r_1r_2}}$.

(β) $\sin\dfrac{A}{2}=\dfrac{r_1}{\sqrt{(r_1+r_2)(r_1+r_3)}}$.

(γ) $\sin A=2r_1\dfrac{\sqrt{r_2r_3+r_3r_1+r_1r_2}}{(r_1+r_2)(r_1+r_3)}$.

(δ) $\cos A=\dfrac{2R+r-r_1}{2R}$.

Show that

9. $$16R^2rr_1r_2r_3=a^2b^2c^2.$$

10. $$R=\tfrac{1}{4}\frac{(r_2+r_3)(r_3+r_1)(r_1+r_2)}{r_2r_3+r_3r_1+r_1r_2}.$$

11. If the diameter of an escribed circle is equal to the perimeter of the triangle, the triangle is right-angled.

12. In an equilateral triangle whose side is a, show that

$$r=\frac{a}{2\sqrt{3}},\qquad R=\frac{a}{\sqrt{3}}.$$

13. If $b=c$ prove that

$$rr_1=\frac{a^2}{4}.$$

14. If the escribed circle which touches a is equal to the circum-circle, prove that

$$\cos A = \cos B + \cos C.$$

15. Prove that

$$r_1 (r_2 + r_3) \operatorname{cosec} A = r_2 (r_3 + r_1) \operatorname{cosec} B = r_3 (r_1 + r_2) \operatorname{cosec} C.$$

16. In the triangle HJK show that the sides are

$$2 (s - a) \sin \frac{A}{2}, \quad 2 (s - b) \sin \frac{B}{2}, \quad 2 (s - c) \sin \frac{C}{2};$$

and that the angles are

$$90^\circ - \frac{A}{2}, \quad 90^\circ - \frac{B}{2}, \quad 90^\circ - \frac{C}{2}.$$

17. Show that the area of the triangle HJK is equal to

$$2S \frac{(s-a)(s-b)(s-c)}{abc}, \text{ and its circum-radius to } \frac{S}{s}.$$

18. Prove that

$$IA = r \operatorname{cosec} \frac{A}{2}, \quad I_1A = r_1 \operatorname{cosec} \frac{A}{2},$$

$$II_1 = a \sec \frac{A}{2}, \quad I_2I_3 = a \operatorname{cosec} \frac{A}{2}.$$

19. Show that the area of $I_1I_2I_3$ is $\dfrac{abc}{2r}$.

20. Show that the radius of the circle round any one of the four triangles formed by joining the centres of the inscribed and escribed circles is double of R.

21. Prove that the areas $I_1I_2I_3$, I_2I_3I, I_3I_1I, I_1I_2I, are inversely as r, r_1, r_2, r_3.

22. If $d_1d_2d_3$ be the distances of I from the angular points of a triangle, show that

$$\frac{d_1d_2d_3}{abc} = \frac{r}{s}.$$

23. Show that the radius of the circle circumscribed to the triangle BOC is $\frac{1}{2}R\sec A$.

24. The radius of the circum-circle of IBC is

$$\tfrac{1}{2}a\sec\frac{A}{2}.$$

25. In the ambiguous case, where a, b, A are given, show that

$$\left(\frac{c_1}{r_1}-\cot\frac{A}{2}\right)\left(\frac{c_2}{r_2}-\cot\frac{A}{2}\right)=1,$$

$$r_1r_2=b(b-a)\sin^2\frac{A}{2},$$

r_1, r_2 being the radii of the *inscribed* circles of the two triangles.

26. Prove that the circum-circles of the two triangles in the ambiguous case are equal in magnitude; also that the distance between their centres is $(b^2\operatorname{cosec}^2 B-a^2)^{\frac{1}{2}}$; a, b, B being given.

27. Show that

$$IA\,.\,II_1=IB\,.\,II_2=IC\,.\,II_3=4Rr.$$

28. If α, α_1, α_2, α_3 are the distances of the centres of the inscribed and escribed circles from A, and p is the perpendicular from A on BC, prove that

$$\alpha\alpha_1\alpha_2\alpha_3=4R^2p^2,$$

$$\alpha^2+\alpha_1^2+\alpha_2^2+\alpha_3^2=16R^2,$$

$$\frac{1}{\alpha^2}+\frac{1}{\alpha_1^2}+\frac{1}{\alpha_2^2}+\frac{1}{\alpha_3^2}=\frac{4}{p^2}.$$

119. The medians.

The lines AD, BE, CF, joining the angular points of a triangle to the middle points of the opposite sides, are called the *medians*.

The length of AD is given by the well-known geometrical theorem

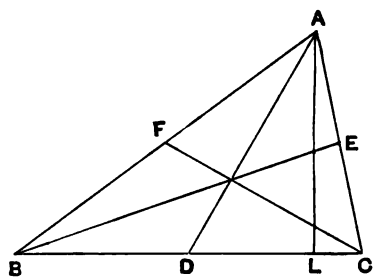
$$AB^2 + AC^2 = 2\,(AD^2 + BD^2)\,;$$

thus the squares of their lengths are given by

$$m_1^2 = \tfrac{1}{2}b^2 + \tfrac{1}{2}c^2 - \tfrac{1}{4}a^2,\quad m_2^2 = \tfrac{1}{2}c^2 + \tfrac{1}{2}a^2 - \tfrac{1}{4}b^2,\quad m_3^2 = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 - \tfrac{1}{4}c^2 \qquad \text{............(11).}$$

Let M_1 denote the angle ADC, then

$$\cot M_1 = \frac{DL}{AL}, \text{ where } AL \text{ is perpendicular to } BC.$$



Also $DL = \frac{1}{2}(BL - CL)$, therefore

$$\cot M_1 = \tfrac{1}{2}\,\frac{BL - CL}{AL} = \tfrac{1}{2}\,(\cot B - \cot C) \;\ldots\; (12).$$

The point G where the medians intersect one another is called the *centroid* of the triangle. It is well known that G divides each of the medians in the ratio $2:1$.

EXAMPLES.

1. If the angles BAD, CAD are θ and ϕ show that $\frac{\sin\theta}{b}=\frac{\sin\phi}{c}$, and deduce that $\tan\frac{\theta-\phi}{2}=\frac{b-c}{b+c}\tan\frac{A}{2}$.

2. Show that $m_1^2=bc\cos A+\frac{1}{4}a^2$, and $\sin BGC=\frac{3}{2}\frac{S}{m_2m_3}$.

3. Prove that the radius of the circle circumscribing the triangle BGC is equal to $\frac{am_2m_3}{3S}$.

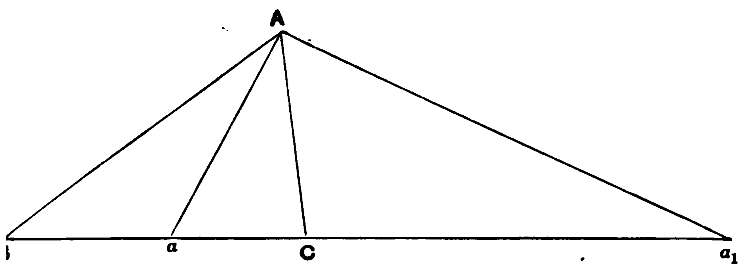
120. The bisectors of the angles.

Let a and a_1 be the points where the internal and external bisectors of the angle A meet the opposite side BC. Let f and f' be the lengths of Aa and Aa_1 respectively.

To find the positions of a and a_1 we have by Euc. VI. 3

$$\frac{Ba}{Ca}=\frac{BA}{CA}=\frac{c}{b}.$$

And $Ba+Ca=a$; hence $Ba=\frac{ac}{b+c}$, $\quad Ca=\frac{ab}{b+c}$.



In the same way, by Euc. VI. 3 A,

$$\frac{Ba_1}{Ca_1}=\frac{BA}{CA}=\frac{c}{b}.$$

And $Ba_1 - Ca_1 = a$; hence

$$Ba_1 = \frac{ac}{c-b}, \quad Ca_1 = \frac{ab}{c-b}.$$

To find the lengths f, f' we have, since

$$2 \triangle ACa + 2 \triangle ABa = 2S,$$

$$fb \sin\frac{A}{2} + fc \sin\frac{A}{2} = 2S,$$

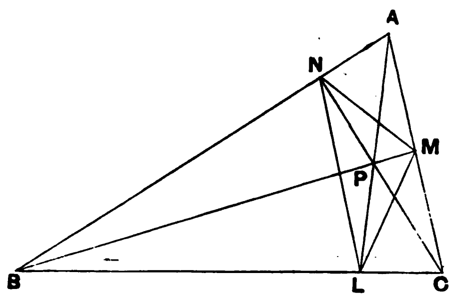
$$\therefore \quad f = 2S \div \left\{(b+c)\sin\frac{A}{2}\right\}.$$

Similarly $\quad f' = 2S \div \left\{(c-b)\cos\frac{A}{2}\right\}.$

Thus $\quad f = \frac{2bc}{b+c}\cos\frac{A}{2}, \quad f' = \frac{2bc}{c-b}\sin\frac{A}{2}$ ....(13).

121. The pedal triangle.

The triangle LMN formed by joining the feet of the perpendiculars AL, BM, CN from A, B, C on the opposite sides, is called the *pedal triangle* of ABC. The perpendiculars intersect in the same point P called the *orthocentre*.



We can find the lengths of the sides of the triangle LMN as follows:

$$MN^2 = AN^2 + AM^2 - 2AM \cdot AN \cos A.$$

And $AN = b \cos A$, $AM = c \cos A$, therefore

$$MN^2 = (b^2 + c^2 - 2bc \cos A) \cos^2 A = a^2 \cos^2 A.$$

Thus the sides of the pedal triangle are

$$a \cos A, \quad b \cos B, \quad c \cos C.$$

Since a circle can be described round the figure $PLMC$, because the angles at M and L are right angles, therefore

$$\angle PLM = \angle PCM = 90° - A.$$

For a similar reason $\angle PLN = PBN = 90° - A$.

Hence the whole $\angle NLM = 180° - 2A$; thus the angles of the pedal triangle are

$$180° - 2A, \quad 180° - 2B, \quad 180° - 2C.$$

Observe that P is the centre of the inscribed circle of the triangle LMN.

The pedal triangle of $I_1 I_2 I_3$ is ABC.

EXAMPLES.

1. Prove that the area of the pedal triangle is

$$2S \cos A \cos B \cos C.$$

2. In the triangle LMN show that

(i) A, B, C are the centres of its escribed circles.

(ii) Radius of circumscribed circle is $\frac{R}{2}$.

(iii) Radius of inscribed circle is $2R \cos A \cos B \cos C$.

(iv) Radii of escribed circles are

$$2R \cos A \sin B \sin C, \text{ \&c.}$$

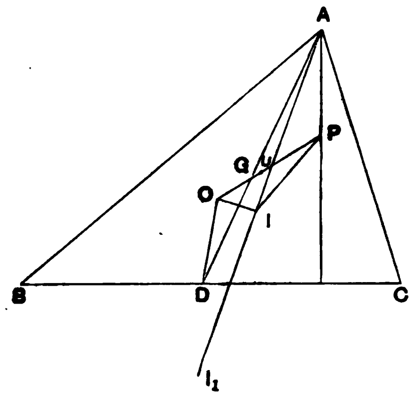
122. The distances between special points.

As before let P be the orthocentre, O the centre of the circum-circle, I of the in-circle, I_1 of one of the escribed

circles, G the centroid, and let U be the centre of the nine-point circle.

We know that O, G, P all lie in one straight line, and $PG = 2OG$; the point U also is the middle point of OP.

Also $OA = R$; and AP being twice the perpendicular OD, from O on BC, is $2R \cos A$.



$$AI = r \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{B}{2} \sin \frac{C}{2}, \text{ from (6)},$$

$$AI_1 = r_1 \operatorname{cosec} \frac{A}{2} = 4R \cos \frac{B}{2} \cos \frac{C}{2}, \text{ from (9)},$$

$$\angle IAP = \frac{A}{2} - \angle PAC = \frac{A}{2} - (90^\circ - C) = \tfrac{1}{2}(C - B),$$

$$\angle IAO = \angle OAC - \frac{A}{2} = 90^\circ - B - \frac{A}{2} = \tfrac{1}{2}(C - B).$$

We can now find expressions for the distances of the points O, I, P, I_1, U, from one another.

(i) *To find* $OI = \delta$.

We have $\delta^2 = AO^2 + AI^2 - 2AO \,.\, AI \cos OAI,$

hence

$$\delta^2 = R^2 \{1 + 16 \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C - 8 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(B - C)\},$$

or

$$\delta^2 = R^2 \{1 - 8 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C\,(-2 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C + \cos \tfrac{1}{2}(B - C))\}$$
$$= R^2 (1 - 8 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C),$$

thus $$\delta^2 = R^2 - 2Rr \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14).$$

(ii) *To find* $OI_1 = \delta_1$.

We have $\delta_1^2 = AO^2 + AI_1^2 - 2AO \,.\, AI_1 \cos OAI_1$,

hence

$$\delta_1^2 = R^2 \{1 + 16 \cos^2 \tfrac{1}{2}B \cos^2 \tfrac{1}{2}C - 8 \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \cos \tfrac{1}{2}(B - C)\},$$

or

$$\delta_1^2 = R^2 \{1 + 8 \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C\,(2 \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C - \cos \tfrac{1}{2}(B - C))\}$$
$$= R^2 \{1 + 8 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C\},$$

which gives $$\delta_1^2 = R^2 + 2Rr_1 \quad\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

(iii) *To find* OP.

From the triangle OAP we have

$$OP^2 = OA^2 + AP^2 - 2OA \,.\, AP \cos OAP$$
$$= R^2 \{1 + 4 \cos^2 A - 4 \cos A \cos (B - C)\},$$

which gives

$$OP^2 = R^2 (1 - 8 \cos A \cos B \cos C) \;\ldots\ldots\; (16).$$

(iv) *To find* IP.

$$IP^2 = AP^2 + AI^2 - 2AP \,.\, AI \cos IAP$$
$$= 4R^2 \cos^2 A + 16R^2 \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C$$
$$- 16R^2 \cos A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(B - C),$$

hence

$$IP^2 = 4R^2 \{\cos^2 A + (1 - \cos B)(1 - \cos C)$$
$$- \cos A \sin B \sin C - \cos A\,(1 - \cos B)(1 - \cos C)\},$$

or

$$IP^2 = 4R^2\{(1-\cos A)(1-\cos B)(1-\cos C) - \cos A \cos B \cos C\}$$
$$= 2r^2 - 4R^2 \cos A \cos B \cos C.$$

(v) *To find IU.*

We have $$IU^2 = \tfrac{1}{2}IP^2 + \tfrac{1}{2}IO^2 - \tfrac{1}{4}OP^2,$$

hence

$$IU^2 = r^2 - 2R^2 \cos A \cos B \cos C + \tfrac{1}{2}R^2 - Rr$$
$$- \tfrac{1}{4}R^2(1 - 8\cos A \cos B \cos C) = r^2 + \tfrac{1}{4}R^2 - Rr,$$

thus $$IU^2 = (r - \tfrac{1}{2}R)^2.$$

From this it follows that $IU = \frac{1}{2}R - r$; similarly it may be shown that

$$I_1U = r_1 + \tfrac{1}{2}R; \quad I_2U = r_2 + \tfrac{1}{2}R; \quad I_3U = r_3 + \tfrac{1}{2}R.$$

Now $\frac{1}{2}R$ is the radius of the nine-point circle; thus since $IU = \frac{1}{2}R - r$, the distance between the centres of the in-circle and the nine-point circle, is equal to the difference of their radii, *therefore these circles touch.*

For a similar reason the nine-point circle *touches each escribed circle.* This is called Feuerbach's Theorem.

EXAMPLES XXXVII.

Prove the following results.

1. $m_1^2 + m_2^2 + m_3^2 = \frac{3}{4}(a^2 + b^2 + c^2)$.

2. $\dfrac{b-c}{r_1} + \dfrac{c-a}{r_2} + \dfrac{a-b}{r_3} = 0$.

3. $2R = (a\cos A + b\cos B + c\cos C) \div 2\sin A \sin B \sin C$.

4. $\dfrac{AI}{AI_1} + \dfrac{BI}{BI_1} + \dfrac{CI}{CI_1} = 1$.

5. If the sides of a triangle are 17, 25, 26 feet, show that:

(i) its area is 204 sq. feet.

(ii) its circum-radius is 13 feet, $6\frac{1}{2}$ inches.

(iii) its in-radius is 6 feet.

(iv) radii of escribed circles are 12 feet, $25\frac{1}{2}$ feet, $22\frac{2}{3}$ feet.

6. The sides of a triangle are a and $2a$, and they include a right angle, show that if r_1 and r_2 are radii of escribed circles touching them $r_1r_2 = a^2$.

7. In an isosceles triangle

$$IP = \frac{a}{2}\frac{\cos\dfrac{3B}{2}}{\sin B\cos\dfrac{B}{2}},$$

where a is the base.

8. If D and α are the points where the bisectors of the side a and the angle A meet that side, show that

$$D\alpha = \frac{a}{2}\,\frac{b \sim c}{b+c}.$$

9. If ρ_1, ρ_2, ρ_3 are the radii of the circles circumscribing the triangles AJK, BKH, CHJ, where H, J, K are the points of contact of the inscribed circle, show that

$$2\rho_1\rho_2\rho_3 = r^2R.$$

10. If p_1, p_2, p_3 are the perpendiculars from the vertices on the opposite sides, and q_1, q_2, q_3 from the centre of the circum-circle, prove that

$$\frac{r-q_1}{p_1} + \frac{r-q_2}{p_2} + \frac{r-q_3}{p_3} = 0.$$

11. The in-circle touches AB in K, and the circle whose centre is I_1 touches AB in K', if AK, KB, and BK' are in geometrical progression, show that

$$\tan^2\frac{B}{2}=\tan\frac{A}{2}\tan\frac{C}{2}.$$

12. If two circles touch externally, and θ is the inclination of their common tangent to the line joining their centres, show that

$$\sin\theta=\frac{a\sim b}{a+b}.$$

13. If α, β, γ be the points where AI, BI, CI meet the sides,

$$\frac{AI\,.\,BI\,.\,CI}{\alpha I\,.\,\beta I\,.\,\gamma I}=\frac{(b+c)\,(c+a)\,(a+b)}{abc}.$$

14. Show that the radius of the circle which touches the sides b and c and the inscribed circle is

$$r\,.\,\frac{1-\sin\dfrac{A}{2}}{1+\sin\dfrac{A}{2}}.$$

15. Show that the area of the triangle whose sides are $b+c$, $c+a$, $a+b$, is $\sqrt{(a+b+c)\,abc}$; and that if ρ is its radius, $\rho^2=2Rr$.

16. If d_1, d_2, d_3 are the distances of any point on the circum-circle from the vertices of the triangle ABC

$$\frac{a}{d_1}\pm\frac{b}{d_2}\pm\frac{c}{d_3}=\frac{abc}{d_1d_2d_3}.$$

17. If $\sin^2\dfrac{A}{2}+\sin^2\dfrac{B}{2}=\frac{1}{4}$, then will $(2r-R)^2=4R^2-c^2$.

18. If P is any point on the circum-circle

$$AP^2 \sin 2A + BP^2 \sin 2B + CP^2 \sin 2C$$
$$= 2R^2 (\sin 2A + \sin 2B + \sin 2C).$$

19. Show that the perimeter of the pedal triangle is

$$4R \sin A \sin B \sin C.$$

20. The line joining the angle A of a triangle to D the middle point of the opposite side makes angles θ and ϕ with AB, AC respectively, show that if $AD = m_1$,

$$2m_1 \cos \frac{\theta - \phi}{2} = (b + c) \cos \frac{A}{2}.$$

21. If $4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = 1$, then will $S = Rr$.

22. The sides of the triangle formed by joining the centres of the circles BOC, COA, AOB are as

$$\sin 2A : \sin 2B : \sin 2C.$$

23. If AL, BM, CN meet the circum-circle in L', M', N', show that

$$\frac{AL'}{AL} + \frac{BM'}{BM} + \frac{CN'}{CN} = 4.$$

24. If l, m, n be the distances from the angular points of a triangle to the points of contact of the inscribed circle with the sides, the radius of the inscribed circle is

$$\left(\frac{lmn}{l + m + n}\right)^{\frac{1}{2}}.$$

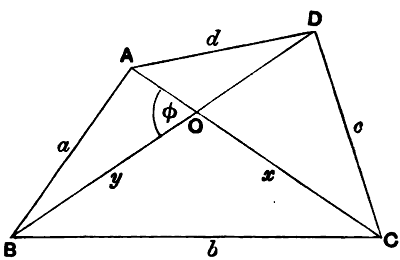
25. If r' is the radius of the circum-circle of the triangle ABO, where O is the centre of the circum-circle of ABC, the distance of the common chord of the circle ABO and the in-circle from O is $r^2 \div 2r'$ where r is the radius of the in-circle of ABC.

26. If the centre of the in-circle be equidistant from the centres of the circum-circle and the orthocentre, prove that one angle of the triangle is 60°.

CHAPTER XVIII.

PROPERTIES OF QUADRILATERALS AND REGULAR POLYGONS.

123. Let $ABCD$ be a convex quadrilateral; denote the sides AB, BC, CD, DA by a, b, c, d respectively, and the diagonals AC, BD by x and y; also let $\angle A + \angle C = 2\alpha$, and let ϕ be the angle between the diagonals.



We shall find an expression for the area S of the quadrilateral in terms of a, b, c, d and α.

We have

$$y^2 = a^2 + d^2 - 2ad \cos A = b^2 + c^2 - 2bc \cos C,$$

therefore

$$\tfrac{1}{2}(a^2 + d^2 - b^2 - c^2) = ad \cos A - bc \cos C,$$

also $$2S = ad \sin A + bc \sin C.$$

Square and add each side of these equations, we get

$$\tfrac{1}{4}(a^2+d^2-b^2-c^2)^2+4S^2$$
$$=a^2d^2+b^2c^2-2abcd(\cos A\cos C-\sin A\sin C),$$

and since

$$\cos A\cos C-\sin A\sin C=\cos 2\alpha=2\cos^2\alpha-1,$$

we have

$$\tfrac{1}{4}(a^2+d^2-b^2-c^2)^2+4S^2=a^2d^2+b^2c^2+2abcd-4abcd\cos^2\alpha,$$

or $\quad 16S^2=4(ad+bc)^2-(a^2+d^2-b^2-c^2)^2-16abcd\cos^2\alpha;$

now putting $\qquad a+b+c+d=2s,$

we see that

$$4(ad+bc)^2-(a^2+d^2-b^2-c^2)^2$$
$$=\{2(ad+bc)+a^2+d^2-b^2-c^2\}\{2(ad+bc)-a^2-d^2+b^2+c^2\}$$
$$=\{(a+d)^2-(b-c)^2\}\{(b+c)^2-(a-d)^2\}$$
$$=16(s-c)(s-b)(s-d)(s-a),$$

therefore

$$S^2=(s-a)(s-b)(s-c)(s-d)-abcd\cos^2\alpha\ldots\ldots(1).$$

124. In the case of a *cyclic* quadrilateral, that is a quadrilateral inscribable in a circle, we have $2\alpha=\pi$, thus

$$S^2=(s-a)(s-b)(s-c)(s-d)\ldots\ldots\ldots\ldots(2).$$

If the quadrilateral can have a circle inscribed in it we have $a+c=b+d=s$, therefore

$$(s-a)(s-b)(s-c)(s-d)=abcd,$$

hence $\qquad S^2=abcd\sin^2\alpha\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$

If the quadrilateral is capable of having circles both circumscribed and inscribed, since then $\alpha=\dfrac{\pi}{2}$,

$$S=\sqrt{abcd}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

125. We can easily find the lengths of x and y in a cyclic quadrilateral as follows,

$$x^2 = a^2 + b^2 - 2ab \cos B,$$

$$x^2 = c^2 + d^2 - 2cd \cos D,$$

hence $$x^2\left(\frac{1}{ab} + \frac{1}{cd}\right) = \frac{a^2 + b^2}{ab} + \frac{c^2 + d^2}{cd},$$

since $$\cos B + \cos D = 0,$$

hence $$x^2 = \frac{(ac + bd)(ad + bc)}{ab + cd}.$$

Similarly it may be shown that

$$y^2 = \frac{(ac + bd)(ab + cd)}{ad + bc}.$$

Thus, $$xy = ac + bd, \quad \frac{x}{y} = \frac{ad + bc}{ab + cd}.$$

126. Expressions for the area of any quadrilateral may be found in terms of x, y, and ϕ, as follows:

$$\triangle ABD = \tfrac{1}{2} AO \,.\, BO \sin\phi + \tfrac{1}{2} AO \,.\, DO \sin(180 - \phi)$$
$$= \tfrac{1}{2} AO \sin\phi\,(BO + DO)$$
$$= \tfrac{1}{2} AO \,.\, y \sin\phi.$$

Similarly $$\triangle BCD = \tfrac{1}{2} CO \,.\, y \sin\phi,$$
hence
$$S = \triangle ABD + \triangle ACD = \tfrac{1}{2} y \sin\phi\,(AO + CO)$$
or $$S = \tfrac{1}{2}\, xy \sin\phi \ldots\ldots\ldots\ldots\ldots (5).$$

Again

$$2OA \,.\, OB \cos\phi = OA^2 + OB^2 - a^2,$$

$$2OC \,.\, OD \cos\phi = OC^2 + OD^2 - c^2,$$

$$2OA \,.\, OD \cos\phi = d^2 - OA^2 - OD^2,$$

$$2OB \,.\, OC \cos\phi = b^2 - OB^2 - OC^2,$$

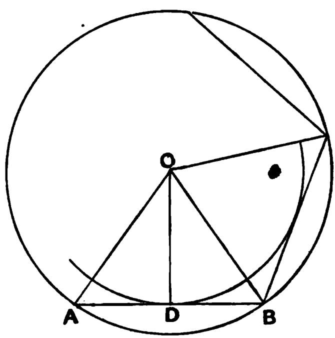
hence adding each side of the four equations,

$$2xy \cos \phi = b^2 + d^2 - a^2 - c^2,$$

therefore
$$S = \tfrac{1}{4}(b^2 + d^2 - a^2 - c^2) \tan \phi \ \ldots\ldots\ldots \ (6).$$

127. Regular polygons.

Let O be the centre of the circles circumscribed about and inscribed in a regular polygon of n sides. Let R, r be the radii of the former and the latter circle, and let a be the length of a side of the polygon.



If AB be a side of the polygon, and D its point of contact with the inscribed circle, the angle AOB is $\frac{2\pi}{n}$, and the angle AOD is $\frac{\pi}{n}$; thus

$$a = 2R \sin \frac{\pi}{n} = 2r \tan \frac{\pi}{n} \ \ldots\ldots\ldots\ldots\ldots \ (7)$$

The area of the triangle OAB is

$$\tfrac{1}{2}R^2 \sin \frac{2\pi}{n}, \text{ or } \tfrac{1}{2}ar,$$

which is equal to $$r^2 \tan \frac{\pi}{n},$$

hence the area of the polygon is

$$\tfrac{1}{2} nR^2 \sin \frac{2\pi}{n}, \text{ or } nr^2 \tan \frac{\pi}{n} \ldots\ldots\ldots\ldots (8).$$

EXAMPLES XXXVIII.

1. If r be the radius of a circle inscribed in a polygon show that

$$r = 2\,\frac{\text{area of polygon}}{\text{sum of sides of polygon}}.$$

2. The length of the side of an equilateral triangle inscribed in a circle is to the side of the inscribed square as $\sqrt{3} : \sqrt{2}$.

3. An equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are as $2:3$.

4. The ratio of the areas of the regular octagons circumscribed to, and inscribed in a circle is equal to

$$2\sqrt{2}\,(\sqrt{2} - 1).$$

5. The sides of a quadrilateral inscribed in a circle are 60, 25, 52, 39; show that the diagonals are 65 and 63, and the area 1764.

6. If a quadrilateral is inscribed in a circle the radius of that circle is $= \frac{1}{4}\left\{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}\right\}^{\frac{1}{2}}$.

7. A quadrilateral is such that a circle can be described about it and a circle inscribed in it, show that the radius of the latter is $\dfrac{2\sqrt{abcd}}{a+b+c+d}$.

8. If a, b, c, d are the perpendiculars from the angles of a quadrilateral upon the diagonals d_1, d_2; show that the sine of the angle between the diagonals is

$$\left\{\frac{(a+c)(b+d)}{d_1 d_2}\right\}^{\frac{1}{2}}.$$

9. The area of a regular inscribed polygon is to that of the circumscribing polygon of the same number of sides as $3:4$. Find the number of sides.

10. If a triangle be formed with sides of the regular hexagon, pentagon and decagon inscribed in the same circle, the triangle is right-angled.

11. If R, r, are the radii of the circumscribed and inscribed circles of a regular polygon, and R', r' those of the regular polygon of the same area but double the number of sides, show that

$$R' = \sqrt{Rr}, \qquad r' = \sqrt{\frac{r}{2}(R+r)}.$$

12. Find the perimeters of the regular polygons of twelve sides inscribed and circumscribed to a circle of radius unity, and hence show that π lies between $3{\cdot}10$ and $3{\cdot}22$.

13. Show that if a quadrilateral whose sides taken in order are a, b, c, d, be such that a circle can be inscribed in it, the circle is greatest when the quadrilateral can be inscribed in a circle, and that then the square on the radius of this inscribed circle is $\dfrac{abcd}{(a+c)(b+d)}$.

14. The area of a quadrilateral in which a circle can be inscribed is $\sqrt{abcd}\sin\dfrac{A+C}{2}$; also

$$\sqrt{ad}\sin\frac{A}{2} = \sqrt{bc}\sin\frac{C}{2}.$$

15. $ABCD$ is a quadrilateral inscribed in a circle of unit radius; α, β, γ are the angles subtended by AB, BC, CD at the circumference; prove that

$$\text{area of } ABCD = 2\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta).$$

16. A polygon of $2n$ sides of which n are equal to a and n equal to b, is inscribed in a circle, show that

$$\text{radius of circle} = \tfrac{1}{2}\left(a^2 + 2ab\cos\frac{\pi}{n} + b^2\right)^{\frac{1}{2}}\operatorname{cosec}\frac{\pi}{n}.$$

17. A polygon has circles of radius R and r described about and inscribed in it. A new polygon, of which the radius of the inscribed circle is ρ, is formed by joining the points of contact of the original polygon with its inscribed circle, prove that

$$r^2 = R\rho.$$

CHAPTER XIX.

THE INVERSE NOTATION.

128. IT is convenient to have a symbol to denote an angle of which one trigonometrical ratio is known. If an angle θ has for its sine the number a, we have $\sin\theta = a$. This fact is also often stated as follows:

$$\theta = \sin^{-1} a;$$

the meaning of the symbol $\sin^{-1}$ being, "*an angle whose sine is.*" *We must carefully distinguish between* $\sin^{-1} a$ *and* $\dfrac{1}{\sin a}$ *since they are quite different in meaning.*

129. Principal Values.

The meaning of $\sin^{-1} a$ has been defined as *an* angle whose sine is a, and not *the* angle whose sine is a; in Art. 45, it is shown that if α is any angle whose sine is a, then all the angles included in

$$n\pi + (-1)^n \alpha$$

will have a for their sine.

Thus from the equation $\theta = \sin^{-1} a$, we infer that θ is *any one* of the set of angles got by giving all integral values to n in the expression

$$n\pi + (-1)^n \alpha.$$

That angle which lies between $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$ and has its sine equal to a is called the *Principal Value* of $\sin^{-1} a$. This principal value has always the same sign as a. It is often, but not always, assumed that when $\sin^{-1} a$ occurs it has this Principal Value.

In like manner we define $\cos^{-1} b$, $\tan^{-1} c$ as angles whose cosine and tangent are respectively b and c. If β is the smallest positive angle whose cosine is b, the Principal Value of $\cos^{-1} b$ is β. The principal value of $\cos^{-1} b$ always lies between 0 and $\frac{\pi}{2}$, when b is positive, and between $\frac{\pi}{2}$ and π when b is negative. The other values are got from the expression $2n\pi \pm \beta$ by giving different integral values to n; see Art. 43.

If γ is the angle between $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$ whose tangent is c, the Principal Value of $\tan^{-1} c$ is γ.

The expressions $\sin^{-1}a$, $\cos^{-1}b$, $\tan^{-1}c$ are called Inverse Circular Functions to distinguish them from the sine, cosine, &c. which are called the Circular Functions.

There are other Inverse Functions: viz., $\operatorname{cosec}^{-1} a$, $\sec^{-1} a$, $\cot^{-1} a$; which have similar meanings to those of the Inverse Functions already described.

130. The two equations

$$A = \sin^{-1} a, \quad A = \cos^{-1} \sqrt{1 - a^2},$$

express the same fact, for since $A = \sin^{-1} a$, it follows that $\sin A = a$, and therefore

$$\cos A = \sqrt{1 - a^2},$$

or
$$A = \cos^{-1} \sqrt{1 - a^2}.$$

Again, since

$$\tan A = \frac{a}{\sqrt{1-a^2}}, \quad A = \tan^{-1}\frac{a}{\sqrt{1-a^2}}.$$

In these results, the radical must be supposed to have either sign.

131. Relations between Inverse Functions.

We can use the formulae of Chapter V. in order to find relations connecting Inverse Functions of two angles. Take for instance

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

It follows that

$$A + B = \sin^{-1}\{\sin A \cos B + \cos A \sin B\}.$$

Let $\qquad \sin A = a, \text{ or, } A = \sin^{-1} a,$

and $\qquad \sin B = b, \text{ or, } B = \sin^{-1} b,$

then we see that

$$\sin^{-1} a + \sin^{-1} b = \sin^{-1}\{a\sqrt{1-b^2} + b\sqrt{1-a^2}\} \ldots (1).$$

Similarly it is shown that

$$\sin^{-1} a - \sin^{-1} b = \sin^{-1}\{a\sqrt{1-b^2} - b\sqrt{1-a^2}\} \ldots (2).$$

Again from the equation,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$
$$A + B = \cos^{-1}\{\cos A \cos B - \sin A \sin B\}.$$

Let now $\qquad \cos A = a, \quad \cos B = b,$

and we see that

$$\cos^{-1} a + \cos^{-1} b = \cos^{-1}\{ab - \sqrt{1-a^2}\sqrt{1-b^2}\} \ldots (3).$$

In like manner

$$\cos^{-1} a - \cos^{-1} b = \cos^{-1}\{ab + \sqrt{1-a^2}\sqrt{1-b^2}\} \ldots (4).$$

If in the equation

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B},$$

we put $\tan A = a, \quad \tan B = b,$

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1} \frac{a+b}{1-ab} \quad \text{.........(5).}$$

This is the simplest equation connecting Inverse Functions.

Ex. 1. If $A = \sin^{-1} \frac{2}{5}$, show that $\cos A = \frac{\sqrt{21}}{5}$.

Since $\sin A = \frac{2}{5}$ it follows that

$$\cos A = \sqrt{1 - \left(\frac{2}{5}\right)^2} = \frac{\sqrt{21}}{5}.$$

Ex. 2. Show that

$$\sin^{-1} \tfrac{1}{3} = \cos^{-1} \frac{2\sqrt{2}}{3} = \tan^{-1} \frac{1}{2\sqrt{2}}.$$

If $\theta = \sin^{-1} \frac{1}{3}$, then $\sin \theta = \frac{1}{3}$ and $\cos \theta = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$.

Also $\tan \theta = \dfrac{\frac{1}{3}}{\frac{2\sqrt{2}}{3}} = \dfrac{1}{2\sqrt{2}}.$

Ex. 3. Prove that

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

The angle whose sine is x and the angle whose cosine is x are complementary, hence their sum is $\frac{\pi}{2}$.

Ex. 4. To show that

$$\tan^{-1} \tfrac{1}{2} + \tan^{-1} \tfrac{1}{3} = \frac{\pi}{4}.$$

Let $\alpha = \tan^{-1} \frac{1}{2}, \quad \beta = \tan^{-1} \frac{1}{3};$

then $$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1.$$

Hence $\alpha+\beta$ must be equal to $\frac{\pi}{4}$, or,

$$\tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{3}=\frac{\pi}{4}.$$

EXAMPLES XXXIX.

1. If $A=\sin^{-1}\frac{3}{4}$, show that $\cos A=\frac{\sqrt{7}}{4}$.

2. Is there such an angle as $\sin^{-1}2$?

3. Show that $\sin^{-1}\frac{1}{2}=\cos^{-1}\frac{\sqrt{3}}{2}$.

Prove the following:

4. $\sin^{-1}a+\sin^{-1}\sqrt{1-a^2}=90°$.

5. $\sin^{-1}\dfrac{a}{\sqrt{a^2+b^2}}=\tan^{-1}\dfrac{a}{b}$.

6. $\sin^{-1}\frac{1}{2}+\sin^{-1}\dfrac{\sqrt{3}}{2}=\sin^{-1}1$.

7. $2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}$.

8. $\tan^{-1}2+\tan^{-1}\frac{1}{2}=\dfrac{\pi}{2}$.

9. $\sin^{-1}\dfrac{1}{\sqrt{82}}+\sin^{-1}\dfrac{4}{\sqrt{41}}=\dfrac{\pi}{4}$.

10. $\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}=45°$.

11. $\cos^{-1}\sqrt{\dfrac{a-x}{a-b}}=\sin^{-1}\sqrt{\dfrac{x-b}{a-b}}$.

12. $\tan^{-1}\frac{1}{2}=\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}$.

13. $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$.

14. $\sin^{-1}\left(\tan\frac{\pi}{4}\right) = 2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

15. $\tan^{-1}p - \tan^{-1}r = \tan^{-1}\frac{p-q}{1+pq} + \tan^{-1}\frac{q-r}{1+qr}$.

16. If $\sin^{-1}a = \tan^{-1}b,\quad a = \frac{b}{\sqrt{1+b^2}}$.

17. $\cos^{-1}x = \sin^{-1}\sqrt{\frac{1-x}{2}} + \cos^{-1}\sqrt{\frac{1+x}{2}}$.

18. $\tan^{-1}\left(\frac{x\cos\phi}{1-x\sin\phi}\right) - \tan^{-1}\left(\frac{x-\sin\phi}{\cos\phi}\right) = \phi$.

19. $\cos^{-1}\frac{3}{\sqrt{10}} + \sin^{-1}\frac{1}{\sqrt{5}} = 45°$.

20. $\tan^{-1}\frac{120}{169} = 2\sin^{-1}\frac{5}{13}$.

21. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.

22. $\tan^{-1}\frac{2}{3} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{10}{11} = \frac{\pi}{2}$.

23. $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$.

24. If $$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi,$$
then $$x^2 + y^2 + z^2 + 2xyz = 1.$$

25. Solve the equations:
$$\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3},$$
$$\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}.$$

26. If $$a = \tan^{-1}\tfrac{1}{7},\quad \beta = \tan^{-1}\tfrac{1}{3};$$
then shall $$\cos 2a = \sin 4\beta.$$

27. Show that
$$\sin^{-1}\frac{4x(x^2-1)}{(x^2+1)^2} - \cos^{-1}\frac{x^2-1}{x^2+1} = 2\cot^{-1}x.$$

28. $\tan^{-1}\dfrac{2x}{2+x^2+x^4}+\tan^{-1}(x-1)+\tan^{-1}(x+1)$
$$=2\tan^{-1}x.$$

29. $\{\tan(\sin^{-1}x)+\cot(\cos^{-1}x)\}^2=2x\tan\{2(\tan^{-1}x)\}$.

30. Prove that

$$\frac{\pi}{4}=4\tan^{-1}\tfrac{1}{5}-\tan^{-1}\tfrac{1}{239}.$$

CHAPTER XX.

SOLUTION OF TRIGONOMETRICAL EQUATIONS.

132. A TRIGONOMETRICAL equation is one which involves powers of a trigonometrical ratio of an angle, as, for instance,

$$4\sin^2\theta - 3\sin\theta - 1 = 0.$$

Or it may involve several of the trigonometrical ratios, as,

$$\sin^4\theta + \cos^2\theta + 2\cot^2\theta = 5.$$

To solve such an equation as the last, it is best to express everything in terms of one trigonometrical ratio, as follows:

$$\sin^4\theta + 1 - \sin^2\theta + \frac{2(1-\sin^2\theta)}{\sin^2\theta} = 5.$$

The following examples will show the method adopted in order to solve trigonometrical equations.

Ex. 1. Solve $\quad 6\sin^2\theta - 5\sin\theta + 2 = 0.$

Considering the equation as a quadratic in $\sin\theta$ we find the solutions to be

$$\sin\theta = \tfrac{1}{2}, \qquad \sin\theta = \tfrac{1}{3}.$$

Hence θ either equals $\sin^{-1}\frac{1}{2}$, or $\sin^{-1}\frac{1}{3}$.

Take the first solution, *one particular* value of θ for which $\sin\theta$ equals $\frac{1}{2}$, is 30°, or $\frac{\pi}{6}$; hence, by Article 45 the general value of θ is

$$\theta = n\pi + \overline{-1}|^n \frac{\pi}{6}.$$

Similarly if α is the least positive angle whose sine is $\frac{1}{3}$, the general value of θ corresponding to the second solution is

$$\theta = n\pi + \overline{-1}|^n \alpha.$$

The solution of a trigonometrical equation thus leads to an *infinite number* of values for the angle.

Ex. 2. Solve $8\sin^2\theta + 6\cos\theta - 9 = 0.$

The equation may be written

$$8(1-\cos^2\theta) + 6\cos\theta - 9 = 0,$$

or $$8\cos^2\theta - 6\cos\theta + 1 = 0.$$

The values of $\cos\theta$ are therefore $\frac{1}{2}$ and $\frac{1}{4}$.

The solution $\cos\theta = \frac{1}{2}$ gives as the general value of θ

$$\theta = 2n\pi \pm \frac{\pi}{3}.$$ See Art. 43.

Also if α is the smallest angle whose cosine is $\frac{1}{4}$, the other general solution is

$$\theta = 2n\pi \pm \alpha.$$

Ex. 3. Solve the equation

$$\sin 3\theta = 3\sin\theta,$$

since $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, for all values of θ, it follows that in the present case

$$\sin^3\theta = 0, \quad \text{or} \quad \theta = n\pi.$$

Ex. 4. Solve the equation

$$\cos\theta + \sin\theta = 1.$$

Dividing each side of the equation by $\sqrt{2}$ it becomes

$$\cos\theta \cdot \frac{1}{\sqrt{2}} + \sin\theta \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}},$$

or,
$$\cos\left(\theta - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}.$$

Hence
$$\theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4},$$

or the values required are

$$2n\pi, \quad 2n\pi + \frac{\pi}{2}.$$

EXAMPLES XL.

Solve the equations:

1. $4\sin^2\theta - 2(\sqrt{3}+1)\sin\theta + \sqrt{3} = 0.$

2. $\sqrt{3}\tan^2\theta - (\sqrt{3}+1)\tan\theta + 1 = 0.$

3. $\tan^2\theta - \tan\theta(\sqrt{3}+1) + \sqrt{3} = 0.$

4. $8\sin^4\theta - 6\sin^2\theta + 1 = 0.$

5. $4\sin^2\theta + 3\operatorname{cosec}^2\theta = 7.$

6. $16\sin^4\theta - 16\sin^2\theta + 3 = 0.$

$2(\cos x + \sec x) = 5.$

8. $2\sin^2\theta + 3\cos\theta - 3 = 0.$

9. $2\sin\theta = \tan\theta.$

10. $\sin 3\theta = 2\sin\theta.$

11. $\tan\theta = 2\sqrt{3}\cos\theta.$

12. $\tan\theta + \cot\theta = \frac{4}{\sqrt{3}}.$

13. $\sin^2\theta + \cos 2\theta = \cos\theta$.

14. $\cos\theta + \tan\theta = \sec\theta$.

15. $\sin\theta + \cos\theta = \dfrac{1}{\sqrt{2}}$.

16. $\cos x + \sqrt{3}\sin x = 1$.

17. $\cos 2\theta + \sin\theta + \cos^2\theta = \frac{7}{4}$.

18. $\cos 2\theta = \cos\theta + \sin\theta$.

19. $4\sin\theta\cos\theta + 1 - 2(\sin\theta + \cos\theta) = 0$.

[Factorize the left side of the equation.]

20. $2\sin^2\theta - \sin\theta = 0$.

21. If $\tan A = \cos A$, show that each is $= \left(\dfrac{\sqrt{5}-1}{2}\right)^{\frac{1}{2}}$.

22. Solve $\cot\theta - \tan\theta = \cot\alpha - \tan\alpha$.

23. Solve the equation $\tan 5\theta = \tan\theta$.

133. The results of Article 56 are often conveniently applied to solve trigonometrical equations; take for instance the equation,

$$\sin 3\theta + \sin 5\theta = 0.$$

This is equivalent to

$$2\sin 4\theta\cos\theta = 0.$$

It follows that *either* $\sin 4\theta = 0$, *or* $\cos\theta = 0$. Hence the general solutions are

$$\theta = \frac{n\pi}{4}, \quad \theta = (2n+1)\frac{\pi}{2}.$$

Again to solve the equation,

$$\sin\theta + \sin 3\theta = \cos\theta.$$

This is the same thing as

$$2\sin 2\theta\cos\theta = \cos\theta.$$

It follows that *either* $\cos\theta = 0$, *or* $\sin 2\theta = \frac{1}{2}$. Thus the solutions are

$$\theta = 2n\pi \pm \frac{\pi}{2}, \quad \theta = n\pi + (-1)^n \frac{\pi}{6}.$$

EXAMPLES. XLI.

Solve the equations:

1. $\sin 7\theta - \sin\theta = \sin 3\theta$.
2. $\sin 3\theta + \sin 5\theta = \sqrt{2}\cos\theta$.
3. $\cos\theta - \cos 5\theta = \sin 2\theta$.
4. $\cos\theta + \cos 2\theta = \sin 3\theta$.
5. $\sin 4\theta - \sin 2\theta = \sqrt{3}\sin\theta$.
6. $\cos(\theta + a) + \cos\theta = 2\cos\frac{a}{2}$.
7. $\sin 50x + \sin 2x = 2\sin 26x$.
8. $\sin 4x - \sin 2x = \sin x$.
9. $\sin 6\theta = 2\sin 4\theta - \sin 2\theta$.
10. $\sin p\theta + \sin q\theta + \sin(p+q)\theta = 0$.
11. $\sin 3\theta + \sin\theta + \cos 4\theta = 1$.
12. $\sin\theta + \sin 2\theta = \sin 3\theta + \sin 4\theta$.
13. $\sin x + \sin 2x = \cos x + \cos 2x$.
14. $\sin\frac{p+1}{2}\theta + \sin\frac{p-1}{2}\theta = \cos\frac{\theta}{2}$.
15. $\cos 2x - \cos 120° = \cos x - \cos 60°$.
16. $\sin(x + 2a) - \sin(2x + a) = \sin\frac{a-x}{2}$.
17. $\sin\theta + \cos\theta = \sin 2\theta + \cos 2\theta$.

134. Equations with Inverse Functions.

A third class of equations is that in which Inverse Functions appear.

To solve the equation,

$$\sin^{-1} x = \sin^{-1} a + \sin^{-1} b.$$

Let $\sin^{-1} a = A$, $\sin^{-1} b = B$; then we have

$$\sin A = a, \quad \sin B = b.$$

Thus

$$\begin{aligned} \sin^{-1} x = A + B, \text{ or } \quad x &= \sin (A + B) \\ &= \sin A \cos B + \cos A \sin B \\ &= a\sqrt{1-b^2} + b\sqrt{1-a^2}. \end{aligned}$$

Sometimes it is better to alter the form of the equation before proceeding to solve, as in the following case:

Solve $\qquad \sin^{-1} x + \tan^{-1} \dfrac{2x}{\sqrt{1-4x^2}} = \dfrac{\pi}{3}.$

Replace $\tan^{-1} \dfrac{2x}{\sqrt{1-4x^2}}$ by its equivalent $\sin^{-1} 2x$; the equation then becomes

$$\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3};$$

or

$$\sin^{-1} x = \frac{\pi}{3} - \sin^{-1} 2x,$$

hence

$$\begin{aligned} x &= \sin\left(\frac{\pi}{3} - \sin^{-1} 2x\right) \\ &= \sin\frac{\pi}{3}\cos(\sin^{-1} 2x) - \cos\frac{\pi}{3}\sin(\sin^{-1} 2x) \\ &= \frac{3\surd}{2}\sqrt{1-4x^2} - \tfrac{1}{2} . 2x. \end{aligned}$$

Thus

$$2x = \frac{\sqrt{3}}{2}\sqrt{1-4x^2},$$

or by squaring,

$$\frac{16x^2}{3} = 1 - 4x^2, \text{ and } x = \frac{\pm\sqrt{3}}{2\sqrt{7}}.$$

EXAMPLES. XLII.

Solve the equations:

1. $\cos^{-1} x = \cos^{-1} \frac{1}{2} + \cos^{-1} \frac{\sqrt{3}}{2}$.

2. $\sin^{-1} x + \sin^{-1} \frac{1}{2} = \sin^{-1} \frac{3}{4}$.

3. $\cos^{-1} \frac{1}{4} - \cos^{-1} x = \cos^{-1} \frac{4}{5}$.

4. $\tan^{-1} x + \tan^{-1} \frac{1}{4} = \tan^{-1} \frac{1}{2}$.

5. $\sin^{-1} x = 2 \sin^{-1} \frac{1}{\sqrt{2}}$.

6. $\sin(\cot^{-1} \frac{1}{2}) = \tan(\cos^{-1} \sqrt{x})$.

7. $\sin^{-1} \frac{12}{13} + \sin^{-1} \frac{12}{x} = \frac{\pi}{2}$.

8. $\cot^{-1} x - \cot^{-1}(x+2) = 15°$.

9. $\tan^{-1} \frac{x+1}{x-1} + \tan^{-1} \frac{x-1}{x} = \tan^{-1}(-7)$.

10. $\operatorname{cosec}^{-1} x = \operatorname{cosec}^{-1} a + \operatorname{cosec}^{-1} b$.

11. $\cos^{-1} \frac{1-x^2}{1+x^2} + \tan^{-1} \frac{2x}{1-x^2} = \frac{4\pi}{3}$.

    [Put $x = \tan\theta$.]

12. $\cos^{-1} \frac{1-a^2}{1+a^2} - \cos^{-1} \frac{1-b^2}{1+b^2} = 2 \tan^{-1} x$.

135. The summation of series.

To sum the series of n terms

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+\overline{n-1}\,\beta);$$

proceed as follows;

$$\cos\alpha = \frac{1}{2\sin\frac{1}{2}\beta}\{\sin(\alpha+\tfrac{1}{2}\beta) - \sin(\alpha-\tfrac{1}{2}\beta)\},$$

$$\cos(\alpha+\beta) = \frac{1}{2\sin\frac{1}{2}\beta}\{\sin(\alpha+\tfrac{3}{2}\beta) - \sin(\alpha+\tfrac{1}{2}\beta)\},$$

$$\cos(\alpha+2\beta) = \frac{1}{2\sin\frac{1}{2}\beta}\{\sin(\alpha+\tfrac{5}{2}\beta) - \sin(\alpha+\tfrac{3}{2}\beta)\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cos(\alpha+\overline{n-1}\beta)$$
$$= \frac{1}{2\sin\frac{1}{2}\beta}\left\{\sin\left(\alpha+\frac{2n-1}{2}\beta\right) - \sin\left(\alpha+\frac{2n-3}{2}\beta\right)\right\}.$$

Now the sum of the left-hand members of these equations is

$$\cos\alpha + \cos(\alpha+\beta) + \ldots + \cos(\alpha+\overline{n-1}\beta).$$

And the sum of the right-hand members is

$$\frac{1}{2\sin\frac{1}{2}\beta}\left\{\sin\left(\alpha+\frac{2n-1}{2}\beta\right) - \sin(\alpha-\tfrac{1}{2}\beta)\right\};$$

since all the other terms cancel out.

Hence

$$\cos\alpha + \cos(\alpha+\beta) + \ldots + \cos(\alpha+\overline{n-1}\beta)$$
$$= \frac{1}{2\sin\frac{1}{2}\beta}\left\{\sin\left(\alpha+\frac{2n-1}{2}\beta\right) - \sin(\alpha-\tfrac{1}{2}\beta)\right\}$$
$$= \frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{1}{2}\beta} \quad\ldots\ldots\ldots\ldots(1).$$

In a similar manner we find that

$$\sin\alpha + \sin(\alpha+\beta) + \ldots + \sin(\alpha+\overline{n-1}\beta)$$
$$= \frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{1}{2}\beta} \quad\ldots\ldots\ldots\ldots(2).$$

We get the last result from the equations

$$\sin \alpha = \frac{1}{2 \sin \frac{\beta}{2}} \{\cos(\alpha - \tfrac{1}{2}\beta) - \cos(\alpha + \tfrac{1}{2}\beta)\},$$

$$\sin 2\alpha = \frac{1}{2 \sin \frac{1}{2}\beta} \{\cos(\alpha + \tfrac{1}{2}\beta) - \cos(\alpha + \tfrac{3}{2}\beta)\}, \text{ \&c.}$$

Or it may be derived from (1) by writing therein $\frac{\pi}{2} + \alpha$ in place of α.

The results (1) and (2) may be applied to solve equations, as follows:

to find θ, when $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \sin 5\theta = 0$.

We get the sum of the left side from (2) by taking in it

$$n = 5, \quad \alpha = \beta = \theta.$$

This gives us

$$\frac{\sin\left(\theta + \frac{5-1}{2}\theta\right) \sin \frac{5\theta}{2}}{\sin \frac{\theta}{2}} = 0;$$

of which the solutions are

$$\begin{cases} \sin 3\theta = 0, \text{ or } \theta = \dfrac{n\pi}{3}, \\ \sin \dfrac{5\theta}{2} = 0, \text{ or } \theta = \dfrac{2n\pi}{5}. \end{cases}$$

136. Particular methods of solution.

The following methods deserve attention.

(i) To solve the equation

$$a \cos\theta + b \sin\theta = c.$$

Divide each side of the equation by $\sqrt{a^2 + b^2}$, and let

$$\frac{a}{\sqrt{a^2 + b^2}} = \cos\alpha, \quad \frac{b}{\sqrt{a^2 + b^2}} = \sin\alpha;$$

giving $$\cos\alpha\cos\theta+\sin\alpha\sin\theta=\frac{c}{\sqrt{a^2+b^2}};$$

whence $$\cos(\theta-\alpha)=\frac{c}{\sqrt{a^2+b^2}},$$

or $$\theta=\alpha+\cos^{-1}\frac{c}{\sqrt{a^2+b^2}}.$$

(ii) To solve the equation
$$a\cos^2\theta+b\sin^2\theta=c.$$

In place of c write
$$c(\sin^2\theta+\cos^2\theta),$$
the equation becomes
$$(b-c)\sin^2\theta=(c-a)\cos^2\theta;$$

whence $$\tan\theta=\pm\sqrt{\frac{c-a}{b-c}}.$$

137. Elimination.

When a quantity which appears in each of two simultaneous equations is got rid of by suitably combining the equations it is said to be *eliminated*.

For instance the result of eliminating y between the equations
$$ax+by=c,$$
$$dx+ey=f$$

is $$\frac{ax-c}{b}-\frac{dx-f}{e}=0.$$

The following are some illustrations of elimination in trigonometrical equations:

Ex. 1. To eliminate θ between the equations
$$\cos\theta+\sin\theta=a,$$
$$\cos\theta-\sin\theta=b.$$

Square each side of the equations and add the results, we get

$$a^2+b^2=(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2,$$

or,
$$a^2+b^2=2.$$

Ex. 2. Eliminate θ between

$$\sin\theta\cos^2\theta=a,$$
$$\sin^2\theta\cos\theta=b.$$

Dividing we get

$$\frac{\sin\theta\cos^2\theta}{\sin^2\theta\cos\theta}=\frac{a}{b},\quad\text{or,}\quad\cot\theta=\frac{a}{b}.$$

Hence $\cos\theta=\dfrac{a}{\sqrt{a^2+b^2}},\quad \sin\theta=\dfrac{b}{\sqrt{a^2+b^2}}.$

Substitute these values in the first equation and we have

$$\frac{b}{\sqrt{a^2+b^2}}\cdot\frac{a^2}{a^2+b^2}=a,$$

or,
$$ab=(a^2+b^2)^{\frac{3}{2}}.$$

Ex. 3. Eliminate θ from the equations

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1,$$

$$x\sin\theta-y\cos\theta=(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{1}{2}}.$$

Square each of the equations and put $\tan\theta=t$, the equations become

$$t^2\left(1-\frac{y^2}{b^2}\right)-2t\frac{xy}{ab}+1-\frac{x^2}{a^2}=0,$$

$$t^2(a^2-x^2)+2txy+b^2-y^2=0.$$

We have to eliminate θ from them. Solving for t and t^2 we get

$$\frac{t^2}{2xy\left(1-\dfrac{x^2}{a^2}+\dfrac{b^2-y^2}{ab}\right)}=\frac{t}{\dfrac{(b^2-y^2)^2}{b^2}-\dfrac{(a^2-x^2)^2}{a^2}}$$

$$=\frac{1}{\dfrac{-2xy\,(a^2-x^2)}{ab}-\dfrac{2xy\,(b^2-y^2)}{b^2}}.$$

Hence

$$\left(\frac{b^2-y^2}{b}+\frac{a^2-x^2}{a}\right)^2\left\{\frac{b^2-y^2}{b}-\frac{a^2-x^2}{a}\right\}^2$$

$$=\frac{-4x^2y^2}{a^3b^3}\{b\,(a^2-x^2)+a\,(b^2-y^2)\}^2,$$

or, $$\left(a+b-\frac{x^2}{a}-\frac{y^2}{b^2}\right)^2\left\{\left(\frac{b^2-y^2}{b}-\frac{a^2-x^2}{a}\right)^2+\frac{4x^2y^2}{ab}\right\}=0.$$

Therefore the result of the elimination is

$$\frac{x^2}{a}+\frac{y^2}{b}=a+b.$$

EXAMPLES. XLIII.

Solve the equations:

1. $\cos x+\cos 7x=\cos 4x.$

2. $\cos\theta+\cos 3\theta=\cos 2\theta+\cos 4\theta.$

3. $\cos(\theta+\alpha)\,.\cos(\theta-\alpha)=\cos 2\alpha.$

4. $3\tan^2 2\theta=1.$

5. $\tan\theta+\tan 2\theta=\tan 3\theta.$

6. $\sin 5\theta+\sin 3\theta+\sqrt{2}\,(\sin\theta+\cos\theta)\cos\theta=0.$

7. $\cos(m-2)\,\theta-\cos m\theta=\sin\theta.$

8. $(1-\tan\theta)(1+\sin 2\theta)=1+\tan\theta$.

9. $\cos\theta\cos 3\theta=\cos 2\theta\cos 4\theta$.

10. $\sin 8\theta=\sin 5\theta\cos 3\theta-\sin 3\theta\cos\theta$.

11. $\sin(\theta-a)=\sin\theta-\sin a$.

12. $\sin 5\theta=\cos 2\theta$.

13. $\tan 3\theta=3\tan\theta$.

14. $\tan(x+a)=\tan(a-x)$.

15. $2\sin 2\theta-4\sin\left(\theta+\frac{\pi}{6}\right)+\sqrt{3}=0$.

16. $3(\sin^4\theta-\cos^4\theta)+4\cos^6\theta=\cos^3 2\theta$.

17. $\cos 3\theta-(\sqrt{3}+1)\cos 2\theta+(\sqrt{3}+3)\cos\theta-\sqrt{3}-1=0$.

18. $\sec^2\theta\operatorname{cosec}^2\theta+2\operatorname{cosec}^2\theta=8$.

19. $\sin\theta\tan\theta=\frac{5}{6}$.

20. $\sin 2\theta-2\sin\theta+2\sqrt{3}\sin^2\frac{\theta}{2}=0$.

21. $2\sin^2 x=\cos^2\frac{3x}{2}$.

22. $\dfrac{\tan(\theta-15^\circ)}{\tan(\theta+15^\circ)}=\frac{1}{3}$.

23. $a\tan(x-A)+b\tan(x+A)=(a-b)\cot A$.

24. $\tan 2x=\tan\frac{2}{x}$.

25. $\begin{cases}\cos(2x+3y)=\frac{1}{2},\\ \cos(3x+2y)=\frac{\sqrt{3}}{2}.\end{cases}$

26. If $\tan(\pi\sin\theta)=\cot(\pi\cos\theta)$,

find $\sin\left(\theta+\frac{\pi}{4}\right)$.

27. Solve the equation

$$\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 0.$$

28. Solve the equation

$$\cos\frac{\theta}{2} + \cos\frac{3\theta}{2} + \cos\frac{5\theta}{2} = 0.$$

29. Eliminate ϕ from the equations

$$\sin\theta = a\cos\phi + b\sin\phi,$$
$$\cos\theta = a\sin\phi - b\cos\phi.$$

30. Eliminate θ and ϕ from

$$m\sin\theta = n\sin\phi,$$
$$m'\cos\theta = n'\cos\phi,$$
$$\theta = 2\phi.$$

31. Eliminate θ from

$$\tan\theta + \sin\theta = m,$$
$$\tan\theta - \sin\theta = n.$$

32. Eliminate θ from

$$(x - a\sin\theta)^2 + (y - a\cos\theta)^2 = (x\cos\theta - y\sin\theta)^2 = a^2.$$

33. Eliminate α from

$$\frac{\cos 3\alpha}{a} + \frac{\cos 2\alpha}{b} + \frac{\cos\alpha}{c} = 0,$$
$$\frac{\cos 2\alpha}{a} + \frac{\cos\alpha}{b} + \frac{1}{2c} = 0.$$

34. At a point P each of the perpendicular straight lines CA, CB subtend an angle α, if $CA = a$, $CB = b$, show that

$$CP = \frac{ab\cos 2\alpha \operatorname{cosec}\alpha}{\sqrt{a^2 + b^2 - 2ab\sin 2\alpha}}.$$

35. If $\quad x\cos\theta = b, \quad x\cos(2\alpha - \theta) = c$; eliminate θ.

36. If
$$a \sin 2\alpha - b \cos 2\alpha = 2c \sin \alpha,$$
$$a \cos 2\alpha + b \sin 2\alpha = c \cos \alpha,$$
prove that
$$4(a^2 + b^2 - c^2)^3 = 27b^2c^4.$$

37. Eliminate θ from the equations
$$x \cos \theta + y \sin \theta = \sin 2\theta,$$
$$y \cos \theta - x \sin \theta = 2 \cos 2\theta.$$

38. Show that the result of eliminating x from the equations
$$\phi = \tan^{-1} x, \quad \theta = \tan^{-1}(x + h) - \tan^{-1} x,$$
is
$$\sin \theta = h \cos \phi \cos (\phi + \theta).$$

39. Eliminate θ from the equations
$$\sin \theta (1 - \cos \theta) = m,$$
$$\cos \theta (1 - \sin \theta) = n.$$

40. If
$$ax \cos (\alpha + \beta) - by \sin (\alpha + \beta) = \frac{a^2 - b^2}{2} \sin (2\alpha + \beta),$$
$$ax \cos (\alpha - \beta) - by \sin (\alpha - \beta) = \frac{a^2 - b^2}{2} \sin (2\alpha - \beta),$$
then
$$a^2x^2 + b^2y^2 = \left(\frac{a^2 - b^2}{2}\right)^2.$$

MISCELLANEOUS EXAMPLES.

1. If the measures of the angles of a triangle referred to 1°, $100'$, $10000''$ as units, be in the proportion of 2, 1, 3, find the angles.

2. Two of the angles of a triangle are $52^\circ\, 53'\, 51''$, $41^{g}\, 22^{\backprime}\, 50$ respectively; find the third angle.

3. An angle is such that the difference of the reciprocals of the number of grades and degrees in it, is equal to its circular measure divided by 2π; find the angle.

4. The angles of a plane quadrilateral are in A.P. and the difference of the greatest and least is a right angle; find the number of degrees in each angle, and also the circular measure.

5. If an arc of ten feet on a circle of eight feet diameter subtend at the centre an angle $143^\circ\, 14'\, 22''$, find the value of π to four decimal places.

6. Find two regular figures such that the number of degrees in an angle of the one is to the number of degrees in an angle of the other as the number of sides in the first is to the number of sides in the second.

7. The apparent angular diameter of the sun is half a degree. A planet is seen to cross its disc in a straight line at a distance from its centre equal to three-fifths of the radius. Prove that the angle subtended at the earth, by the part of the planet's path projected on the sun, is $\pi/450$.

8. Find in degrees the angle whose circular measure is $\frac{7}{8}$.

9. Show that an inch will subtend an angle of 1″ nearly at a distance of 3 miles, the distance being in a direction perpendicular to that in which the inch is measured.

10. Having given $\frac{\sin A}{\sin B}=p$, $\frac{\cos A}{\cos B}=q$, find $\tan A$ and $\tan B$.

11. If $\cos A=\tan B$, $\cos B=\tan C$, $\cos C=\tan A$,

prove that $$\sin A=\sin B=\sin C=2\sin 18°.$$

12. Solve the equations

(1) $\sin\theta+2\cos\theta=1$.

(2) $\frac{\cos\alpha}{\tan\alpha}=\frac{3}{2}$.

(3) $\sqrt{3}\,\text{cosec}^2\theta=4\cot\theta$.

13. Find a general expression for θ, when $\sin^2\theta=\sin^2\alpha$, and also when $\sin\theta=-\cos\theta=1\sqrt{2}$.

14. Two circles of radii a and b touch each other externally; θ is the angle contained by the common tangents to these circles,

prove that $$\sin\theta=\frac{4(a-b)\sqrt{ab}}{(a+b)^2}.$$

15. Prove that

$$\cos\tan^{-1}\sin\cot^{-1}x=\left(\frac{x^2+1}{x^2+2}\right)^{\frac{1}{2}}$$

16. If A, B, and C are in A. P., show that

$$\frac{\sin A-\sin C}{\cos C-\cos A}=\frac{\cos B}{\sin B}.$$

17. Prove that

$$\tan^{-1}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}+\tan^{-1}\sqrt{\frac{3}{2}}=\frac{3\pi}{4}.$$

18. Prove that $\cot^{-1}3+\text{cosec}^{-1}\sqrt{5}=\frac{\pi}{4}$.

19. If $\sin\alpha=m\sin\beta$, and $\tan\alpha=n\tan\beta$,

then $$\cos\alpha=\sqrt{\frac{m^2-1}{n^2-1}}.$$

20. Prove that

$$\sin 48^\circ = \tfrac{1}{8}\{\sqrt{10+2\sqrt{5}}+\sqrt{3}(\sqrt{5}-1)\}.$$

21. If $\cos(\alpha+\beta)\sin(\gamma+\delta)=\cos(\alpha-\beta)\sin(\gamma-\delta)$,

then will $$\tan\delta=\tan\alpha\tan\beta\tan\gamma.$$

22. If $$\frac{m\tan(\alpha-x)}{\cos^2 x}=\frac{n\tan x}{\cos^2(\alpha-x)},$$

then $$x=\frac{1}{2}\left\{\alpha-\tan^{-1}\left(\frac{n-m}{n+m}\tan\alpha\right)\right\}.$$

23. If the fraction $$\frac{a\cos(\theta+\alpha)+b\sin\theta}{a'\sin(\theta+\alpha)+b'\cos\theta}$$

is independent of θ, show that

$$aa'-bb'=(a'b-ab')\sin\alpha.$$

24. Show that

$$A+\tan^{-1}(\cot 2A)=\tan^{-1}(\cot A).$$

25. Prove that

$$\cot^{-1}x-\cot^{-1}(x+1)=\cot^{-1}(1+x+x^2),$$

and thence sum the series

$$\cot^{-1}(1+1+1^2)+\cot^{-1}(1+2+2^2)+\ldots\ldots \text{ to } n \text{ terms}.$$

26. If

$$\tan(\beta+\gamma)=l\tan\alpha,\ \tan(\gamma+\alpha)=m\tan\beta,\ \tan(\alpha+\beta)=n\tan\gamma,$$

and $$(m-n)\tan\alpha+(n-l)\tan\beta+(l-m)\tan\gamma=0,$$

prove that $$\frac{m-n}{l}+\frac{n-l}{m}+\frac{l-m}{n}=0.$$

27. Show that the only solutions of the equation

$\sin 5\theta-3\sin 3\theta+4\sin\theta=0$, are $\theta=n\pi$, $\theta=n\pi+(-1)^n\dfrac{\pi}{4}$.

28. Prove that

$$\cos A\cos B\cos C\cos(A+B+C)+\sin A\sin B\sin C\sin(A+B+C)$$
$$=\cos(B+C)\cos(C+A)\cos(A+B).$$

29. If $\cos(A+B+C)=\cos A\cos B\cos C$, prove that

$$8\sin(B+C)\sin(C+A)\sin(A+B)+\sin 2A\sin 2B\sin 2C=0.$$

30. If $A+B+C=180°$, prove that

$$(\sin A+\sin B+\sin C)(\sin B+\sin C-\sin A)(\sin C+\sin A-\sin B)(\sin A+\sin B-\sin C)=4\sin^2 A\sin^2 B\sin^2 C.$$

31. If α, β are values of θ which satisfy the equation

$$A\cos\theta+B\sin\theta=C,$$

and whose difference is not a multiple of π, then

$$\frac{\cos\frac{1}{2}(\alpha+\beta)}{A}=\frac{\sin\frac{1}{2}(\alpha+\beta)}{B}=\frac{\cos\frac{1}{2}(\alpha-\beta)}{C}.$$

32. Prove that

$$\sin^2 10°+\cos^2 20°-\sin 10°\cos 20°=\sin^2 10°+\cos^2 40°+\sin 10°\cos 40°=\tfrac{3}{4}.$$

33. Prove that

$$\sin\frac{x-y-z}{2}\sin\frac{y-z}{2}+\sin\frac{x+y-z}{2}\sin\frac{y+z}{2}=\sin\frac{x}{2}\sin y.$$

34. Prove that the elimination of θ, ϕ from the equations

$$r\cos(2\theta-\alpha)=m\cos^2\theta,\quad r\cos(2\phi-\alpha)=m\cos^2\phi,$$
$$\tan\theta=\tan\phi+2\sin\beta,$$

gives $$r=m\cos\alpha/(1-\cos^2\alpha\sin^2\beta).$$

35. If $$\tan\theta\tan\phi=\sqrt{\frac{a-b}{a+b}},$$

show that $$(a-b\cos 2\theta)(a-b\cos 2\phi)$$

is independent of θ and ϕ.

36. Put into six simple factors

$$\sin^3(\beta-\gamma)\sin^3(\alpha-\delta)+\sin^3(\gamma-\alpha)\sin^3(\beta-\delta)+\sin^3(\alpha-\beta)\sin^3(\gamma-\delta).$$

37. Prove that

$$\frac{\sin 2\alpha\sin(\beta-\gamma)+\sin 2\beta\sin(\gamma-\alpha)+\sin 2\gamma\sin(\alpha-\beta)}{\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)}=4\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}.$$

38. Prove that

$$\frac{1}{\cos\frac{2\pi}{7}+\cos 2\phi}+\frac{1}{\cos\frac{4\pi}{7}+\cos 2\phi}+\frac{1}{\cos\frac{6\pi}{7}+\cos 2\phi}=\frac{7\tan 7\phi-\tan\phi}{2\sin 2\phi}.$$

39. If $\tan^2\theta=\tan(\theta-\alpha)\tan(\theta-\beta)$

show that $\tan 2\theta=\frac{2\sin^2\alpha\sin^2\beta}{\sin^2(\alpha+\beta)}$.

40. If $3\cos(\beta+\gamma)+4\sin(\beta+\gamma)=\cos(\beta-\gamma)$

and $3\cos(\alpha+\gamma)-2\sin(\alpha+\gamma)=\cos(\alpha-\gamma)$,

prove that $\tan\alpha\tan\beta=-\frac{1}{2}$.

41. Prove that

$$(\tan 7\tfrac{1}{2}^\circ+\tan 37\tfrac{1}{2}^\circ+\tan 67\tfrac{1}{2}^\circ)(\tan 22\tfrac{1}{2}^\circ+\tan 52\tfrac{1}{2}^\circ+\tan 82\tfrac{1}{2}^\circ)=17+8\sqrt{3}.$$

42. Prove that

$$\cos 2\alpha\sin(\beta-\gamma)+\cos 2\beta\sin(\gamma-\alpha)+\cos 2\gamma\sin(\alpha-\beta)$$
$$+\{\sin(\beta-\gamma)\sin(\gamma-\alpha)+\sin(\alpha-\beta)\}\{\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)\}=0.$$

43. Prove that $\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2}$.

44. If $x=\frac{\sin\theta}{\sin(\theta+A)}$, $y=\frac{\sin\phi}{\sin(\phi+B)}$, $z=\frac{\sin\psi}{\sin(\psi+C)}$,

and $\theta+\phi+\psi=A+B+C=\pi$,

show that $(x-yz)\sin A+(y-zx)\sin B+(z-xy)\sin C=0$.

45. Show that $\tan 67^\circ 30'=1+\sqrt{2}$.

46. Solve the equation

$$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}(-2).$$

47. If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$,

then $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$.

48. If $\tan A+\tan 2A=\tan 3A$, show that A must be a multiple of $60°$ or $90°$.

49. If $a=\dfrac{16}{\pi}$, show that

$$\cos 3a+\cos 5a=\cos a\,(\cos 4a+\sin 4a).$$

50. Prove that

$$\sin^2 A+\sin^2 B+\sin^2(A+B)=2\sin A\sin B\sin(A+B)\{\cot A+\cot B=\cot(A+B)\}.$$

51. Prove that $16\cos^6 A-\cos 5A=5\cos A\,(1+2\cos 2A)$.

52. Prove that

$$\operatorname{cosec}(m+n)x\operatorname{cosec} mx\operatorname{cosec} nx-\cot(m+n)x\cot mx\cot nx=\cot mx+\cot nx-\cot(m+n)x.$$

53. Prove that

$$\frac{\cos 3A}{\cos A}-\frac{\cos 6A}{\cos 2A}+\frac{\cos 9A}{\cos 3A}-\frac{\cos 18A}{\cos 6A}=2\{\cos 2A-\cos 4A+\cos 6A-\cos 12A\}.$$

54. If $$\alpha+\beta+\gamma=\tfrac{1}{2}\pi,$$
prove that $\sin^2\alpha+\sin^2\beta+\sin^2\gamma+2\sin\alpha\sin\beta\sin\gamma=1$.

55. Prove that

$$\sin^2(\theta+\alpha)+\sin^2(\theta+\beta)-2\cos(\alpha-\beta)\sin(\theta+\alpha)\sin(\theta+\beta)$$
is independent of θ.

56. If $\sqrt{2}\cos A=\cos B+\cos^2 B$, $\sqrt{2}\sin A=\sin B-\sin^2 B$,
prove that $$\pm\sin(A-B)=\cos 2B=\tfrac{1}{3}.$$

57. If $\dfrac{\cos^4\alpha}{\cos^2\beta}+\dfrac{\sin^4\alpha}{\sin^2\beta}=1$, show that $\dfrac{\cos^4\beta}{\cos^2\alpha}+\dfrac{\sin^4\beta}{\sin^2\alpha}=1$.

58. If $\alpha+\beta+\gamma=\tfrac{1}{4}\pi$, show that

$$(\cos\alpha+\sin\alpha)(\cos\beta+\sin\beta)(\cos\gamma+\sin\gamma)=2(\cos\alpha\cos\beta\cos\gamma+\sin\alpha\sin\beta\sin\gamma).$$

59. If $$\frac{\tan(\alpha+\beta-\gamma)}{\tan(\alpha-\beta+\gamma)}=\frac{\tan\gamma}{\tan\beta},$$
prove that $$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0.$$

60. If $$\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi,$$
show that $$x^2 + y^2 + z^2 + 2xyz = 1.$$

61. Solve the equations :

(1) $\sin\theta + \cos\theta = 1.$

(2) $\sin 5\theta = 16 \sin^5 \theta.$

(3) $\tan 2\theta = 8\cos^2\theta - \cot\theta.$

(4) $2\sin(\theta - \phi) = \sin(\theta + \phi) = 1.$

(5) $\sec 4\theta - \sec 2\theta = 2.$

(6) $2(\sin^4\theta + \cos^4\theta) = 1.$

62. If $2\cos\theta = \sqrt{1 - \sin 2\theta} - \sqrt{1 + \sin 2\theta}$, show that θ must lie between $(8n+5)\dfrac{\pi}{4}$ and $(8n+7)\dfrac{\pi}{4}$.

63. Prove that
$$\tan\tfrac{1}{2}(x+y)\tan\tfrac{1}{2}(x-y) = \frac{\operatorname{cosec} 2x \operatorname{cosec} y - \operatorname{cosec} 2y \operatorname{cosec} x}{\operatorname{cosec} 2x \operatorname{cosec} y + \operatorname{cosec} 2y \operatorname{cosec} x}.$$

64. Show that if $\cot\frac{1}{2}\alpha + \cot\frac{1}{2}\beta = 2\cot\theta$, then
$$\{1 - 2\sec\theta\cos(\alpha - \theta) + \sec^2\theta\}\{1 - 2\sec\theta\cos(\beta - \theta) + \sec^2\theta\} = \tan^4\theta.$$

65. If $A + B + C + D = 360°$, prove that
$$\cos\tfrac{1}{2}A\cos\tfrac{1}{2}D\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C - \cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\sin\tfrac{1}{2}A\sin\tfrac{1}{2}D$$
$$= \sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A+C)\cos\tfrac{1}{2}(A+D).$$

66. Eliminate θ from the equations
$$(a+b)\tan(\theta - \phi) = (a-b)\tan(\theta + \phi), \quad a\cos 2\phi + b\cos 2\theta = c.$$

67. If A, B, C be positive angles whose sum is 180°, prove that $\cos A + \cos B + \cos C > 1$ and $\ngtr 3/2$.

68. If $$\frac{\cos\theta}{\cos\alpha} + \frac{\sin\theta}{\sin\alpha} = \frac{\cos\phi}{\cos\alpha} + \frac{\sin\phi}{\sin\alpha} = 1,$$
prove that $$\frac{\cos\theta\cos\phi}{\cos^2\alpha} + \frac{\sin\theta\sin\phi}{\sin^2\alpha} + 1 = 0.$$

69. If $\cos(y-z) + \cos(z-x) + \cos(x-y) = -3/2$, show that
$$\cos^3(x+\theta) + \cos^3(y+\theta) + \cos^3(z+\theta)$$
$$- 3\cos(x+\theta)\cos(y+\theta)\cos(z+\theta) = 0.$$

70. Solve the equations $\left.\begin{array}{l}\sin^{-1}x-\sin^{-1}y=\frac{2}{3}\pi\\ \cos^{-1}x-\cos^{-1}y=\frac{1}{3}\pi\end{array}\right\}$.

71. If $a\sin\theta+b\cos\theta=a\operatorname{cosec}\theta+b\sec\theta$, show that each expression is equal to $(a^{\frac{2}{3}}-b^{\frac{2}{3}})(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{1}{2}}$.

72. Sum to n terms the series

$$\sin^2 a+\sin^2 2a+\sin^2 3a+\ldots\ldots$$

73. Sum to n terms the series

$$\sin x\sin 2x\sin 3x+\sin 2x\sin 3x\sin 4x+\ldots$$
$$+\sin nx\sin(n+1)x\sin(n+2)x.$$

74. Sum to n terms

$$\tan x+\frac{1}{2}\tan\frac{x}{2}+\frac{1}{4}\tan\frac{x}{4}+\ldots\ldots+\frac{1}{2^{n-1}}\tan\frac{x}{2^{n-1}}.$$

75. Prove that $\sin(\cos\theta)<\cos(\sin\theta)$, for all values of θ.

76. Prove that

$$\frac{2\cos 2^n\theta+1}{2\cos\theta+1}=(2\cos\theta-1)(2\cos 2\theta-1)\ldots(2\cos 2^{n-1}\theta-1).$$

77. In any triangle, show that

$$\cos A+\cos B=\frac{2(a+b)}{c}\sin^2\frac{C}{2}.$$

78. One angle of a triangle is 15°, and another is $2\frac{2}{3}$ of the third; show that the sides opposite to these latter are as $\sqrt{3}:\sqrt{2}$.

79. In the ambiguous case of the solution of triangles, when a, b, A are given, if C and C' are the two values of one of the angles, show that $\tan A=\cot\frac{1}{2}(C+C')$.

80. If p, q, r be the perpendiculars from the angular points on the opposite sides of a triangle, prove that

$$\frac{p^2}{qr}+\frac{q^2}{rp}+\frac{r^2}{pq}=\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}.$$

81. Prove that in any triangle

$$4R\cos C=r+r_1+r_2-r_3.$$

82. A base line 300 feet in length is measured from the foot of a vertical tower and at the end of this line the angular elevation of the top of the tower is observed to be 33° 41′ 24″. Show that the height of the tower is very nearly 200 feet; having given that

$$L\tan 33^\circ 41' = 9{\cdot}8237981, \text{ tabular diff. for } 1' = 2738,$$
$$\log 2 = {\cdot}30103, \quad \log 3 = {\cdot}4771213.$$

83. If $a=378$, $b=54$, $C=85^\circ 54' 14''$, find A and B, having given that

$$\log 2 = {\cdot}3010300, \quad \log 3 = {\cdot}4771213,$$
$$L\cot 42^\circ 57' = 10{\cdot}0311040, \text{ diff. for } 1' = 2533,$$
$$L\tan 38^\circ 51' = 9{\cdot}9060431, \text{ diff. for } 1' = 2586.$$

84. In a triangle $a=96$, $b=75$, $A=37^\circ 17' 10''$, find B and C, having given that

$$\log 2 = {\cdot}3010300,$$
$$L\sin 37^\circ 17' = 9{\cdot}7822985, \text{ diff. for } 1' = 1659,$$
$$L\sin 28^\circ 14' = 9{\cdot}6749194, \text{ diff. for } 1' = 2352.$$

85. The sides of a triangle are 5, 6, 7; find the greatest angle, having given that

$$L\tan 39^\circ 13' = 9{\cdot}9117245, \text{ diff.} = 2579.$$

Find the angles of the triangle ABC, having given that

$$C = 30^\circ, \quad \frac{a}{b} = \frac{2}{\sqrt{3}}.$$

86. An observer finds that from the doorstep of his house the angular elevation of the top of a church spire is $5a$, and that from the roof above the doorstep it is $4a$. The height of the roof above the doorstep being h, prove that the height of the spire is

$$h \operatorname{cosec} a \cos 4a \sin 5a.$$

87. If p and q are the perpendiculars from A and B on *any* line through the vertex C of a triangle, prove that

$$a^2p^2 + b^2q^2 - 2abpq\cos C = a^2b^2\sin^2 C.$$

88. If a regular pentagon and a regular decagon have the same perimeters, prove that their areas are as $2 : \sqrt{5}$.

89. If in a triangle $\tan\frac{A}{2}=\frac{5}{6}$, $\tan\frac{B}{2}=\frac{20}{37}$, then $a+c=2b$.

90. From A, the foot of a vertical pole, a line ABC where BC equals a is drawn due east, and the angle subtended by the pole at B is double that subtended at C. A line CD equal to BC is next drawn due south, and it is found that AB subtends at D an angle $\tan^{-1}2$; show that height of pole $=\frac{\sqrt{3}}{2}a$.

91. At the distance of 10 miles from a tower, its top just appeared in the horizon; determine its height, having given the earth's diameter to be 7964 miles.

92. Find the area of a circle traced by a pair of compasses, the length of whose legs is 10 inches, and the angle between them 60°.

93. An equilateral triangle and a regular hexagon have the same perimeter; show that the areas of their inscribed circles are as 4 : 9.

94. The area between 3 circles of radii a, b and c which touch one another is

$$\sqrt{abc(a+b+c)}-\tfrac{1}{2}\{a^2\theta+b^2\phi+c^2\psi\},$$

where $$\frac{2\sqrt{abc(a+b+c)}}{(b+c)(c+a)(a+b)}=\frac{\sin\theta}{b+c}=\frac{\sin\phi}{c+a}=\frac{\sin\psi}{a+b}.$$

95. If AD, BE, CF are the bisectors of the vertical angles of a triangle ABC, the areas of the quadrilaterals

$$BCEF,\ CAFD,\ ABDE$$

are as $$a(b+c),\ b(c+a),\ c(a+b).$$

96. If P, Q, R are the feet of the perpendiculars from the vertices, show that

$$BP+CQ+AR-(CP+AQ+BR)=(b-c)(c-a)(a-b)\frac{a+b+c}{abc}.$$

97. D, E, F being the middle points of the sides, prove that $\cot BAD+\cot ACF+\cot CBE$

$$=\cot CAD+\cot BCF+\cot ABE=3(\cot A+\cot B+\cot C).$$

98. E is the middle point of the side AB of the triangle ABC, AL is perpendicular to BC cutting CE in F, show that

$$AF=\frac{ab\sin C}{a+b\cos C}.$$

99. In an equilateral triangle ABC, of which a side is 8 inches long, a point P is taken in BC 3 inches from B, prove $AP=7$ inches.

100. If the median m_1 makes angles β and γ with AB, AC respectively, $$2m_1\cos\frac{\beta-\gamma}{2}=(b+c)\cos\frac{A}{2}.$$

101. The elevation of a steeple, standing upon a horizontal plane, is observed and at a station a feet nearer to it, its elevation is found to be the *complement* of the former. On advancing in the same direction b feet nearer still the elevation is found to be *double* of the first. Show that the height of the steeple is

$$\sqrt{(a+b)^2-\frac{a^2}{4}}\text{ feet.}$$

102. If in the 3 edges which meet in one angle of a cube 3 points ABC be taken at distances a, b, c from the angle respectively, then the area of the triangle ABC is

$$\tfrac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}.$$

103. A ladder whose length is 30 feet stands against a wall at an angle of 60° with the horizon. At what distance from its top must another of the same length be fastened at an angle of 75° with the horizon so as just to reach a window 48 feet from the ground?

104. On the sides of an equilateral triangle 3 squares are described. Compare the area of the triangle formed by joining the centres of these squares with the area of the equilateral triangle.

105. If a circle be inscribed in a square, and between it and the 4 angular points 4 other circles be described, and so on continually, show that the sum of the perimeters of all these

circles : that of given circle as $2\sqrt{2}-2:1$, and the corresponding areas as $3\sqrt{2}-4:2$.

106. The distance between the centres of 2 wheels is a, and the sum of their radii is c, show that the length of the string which crosses between the wheels and just wraps around them is

$$2\left\{\sqrt{a^2-c^2}+c\cos^{-1}\left(-\frac{c}{a}\right)\right\}.$$

107. The alternate angles of a regular pentagon being joined, show that the area of the interior pentagon thus formed is to that of the original figure as $3+\sqrt{5}:3-\sqrt{5}$.

108. If 2 semicircles be described upon the bounding radii of a quadrant, the circle which touches the 3 circumferences will have its radius : radius of quadrant as $\sqrt{2}-1:3\sqrt{2}-1$.

109. An endless band passes round 2 wheels which can revolve about their centres and the diameter of one wheel is equal to the circumference of the other. Show that the circular measure of the angle described by the smaller wheel while the larger makes one complete revolution is $2\pi^2$.

110. If L, M, N are the feet of the perpendiculars from the vertices of a triangle ABC on the opposite sides and R_a, R_b, R_c, R the radii of the circum-circles of the triangles AMN, BNL, CLM, and ρ the radius of the in-circle of CMN, then

$$R^2=R\rho+R_a^2+R_b^2+R_c^2.$$

111. If BC, CF be the perpendiculars from B and C on the opposite sides, and if FE and BC produced meet in Q, show that

$$2(QE^2-QF^2)=(BQ^2-CQ^2)(\cos 2B+\cos 2C).$$

112. $ABCD$ is any quadrilateral, whose sides taken in order are a, b, c, d; show that the product of the area of the triangle ABC and the square of the tangent from D to the circum-circle of ABC is $abcd\sin\alpha$, where α is the sum of a pair of opposite angles.

113. A quadrilateral whose sides are a, b, c, d is inscribed in

a circle; show that the product of the segments of any chord of the circle drawn through the intersection of the diagonals is

$$\frac{abcd\,(ac+bd)}{(ad+bc)\,(ab+cd)}.$$

114. A man walking on a level plain towards a tower observes at a certain point that the elevation of the top of the tower is 10°, and after going 50 yards nearer to the tower that the elevation is 15°. Having given that

$$L \sin 15° = 9{\cdot}4129962, \qquad \log 25{\cdot}783 = 1{\cdot}4113334,$$
$$L \cos\ 5° = 9{\cdot}9983442, \qquad \log 25{\cdot}784 = 1{\cdot}4113503,$$

find the height of the tower to 4 places of decimals.

115. If one angle of a triangle be 60°, the area $10\sqrt{3}$ and the perimeter 20, find the remaining angles and the sides, having given

$$\log 2 = {\cdot}3010300, \qquad L \sin 38°\,12' = 9{\cdot}7912754,$$
$$\log 3 = {\cdot}4771213$$
$$\log 7 = {\cdot}8450980, \qquad \text{diff. for } 1' = 1605.$$

116. Two men A and B stand at a distance a apart on a line running E. and W. At midday when the altitude of the sun is α, A observes the end of B's shadow and B observes the end of A's shadow. The depression of A's line of sight is β and that of B's line of sight is γ. Prove that A's height is

$$a\left\{\frac{\cot^2\alpha + \cot^2\gamma}{\cot^2\beta\cot^2\gamma - \cot^4\alpha}\right\}^{\frac{1}{2}}.$$

117. $ACBP$ is a quadrilateral such that $A\hat{P}B\ (2\beta)$ is bisected by the diagonal CP; $CA = a$, $CB = b$, $A\hat{C}B = \alpha$; show that

$$CP = \frac{ab}{\sin\beta}\,\frac{\sin(\alpha + 2\beta)}{\sqrt{a^2 + b^2 + 2ab\cos(\alpha + 2\beta)}}.$$

118. A person standing between 2 towers observes that they subtend angles each $= \alpha$, and on walking a feet along a line inclined at an angle γ to the line joining the towers he finds that they subtend angles each $= \beta$.

Show that the heights of the towers are the roots of the equation $x^2(\cot^2\beta - \cot^2\alpha) - 2xa\sqrt{\cot^2\beta - \cot^2\alpha\sin^2\gamma} + a^2 = 0$.

119. Equilateral triangles DBC, $D'BC$ are described on the side BC of the triangle ABC; prove that

$$AD^2+AD'^2=a^2+b^2+c^2.$$

120. Find to 3 places of decimals the length of the side of a regular polygon of 12 sides, circumscribed to a circle of unit radius.

121. ABC is an isosceles triangle right angled at C; D is the middle point of AC. Prove that DB divides B into 2 parts whose cotangents are as 2 : 3.

122. If the sides of a triangle are in A. P. and its area is $\frac{3}{5}$ of the area of an equilateral triangle of the same perimeter, show that its sides are in the ratio 3 : 5 : 7.

123. A person ascends a mountain by a direct course, the inclination of his path to the horizon being at first α and afterwards changing to β, which continues to the summit. Given that the mountain is a feet high and the angle of depression of the starting point as observed from the summit is γ, prove that the length of the ascent is

$$\frac{a\cos\left\{\frac{\alpha+\beta}{2}-\gamma\right\}}{\sin\gamma\cos\frac{\beta-\alpha}{2}}.$$

124. If a quadrilateral $ABCD$ is inscribed in a circle, show that $\tan\frac{A}{2}=\sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}}$, where $2s=a+b+c+d$.

EXAMINATION PAPERS.

I.

1. Define a *unit*. What is meant by saying that the *measure* of a quantity is m?

If the unit of angular measure were 6°, what would be the measure of a right angle?

2. Find the number of degrees in 10 grades, and the number of grades in 15 degrees.

3. If the radius of a circle is 7 feet, find the length of the circumference. $[\pi = \frac{22}{7}]$.

4. Find the trigonometrical ratios of 60° and 45°; also the values of $\sin 225^\circ$, $\cos 120^\circ$, $\tan 330^\circ$.

5. Give a geometrical construction for the angle whose cosine is a given number a.

6. A man 6 feet high standing on the bank of a river observes that the depression of a point on the other bank, immediately opposite, is 10°; find the breadth of the river, having given that $\tan 10^\circ = \cdot 1763270$.

7. Show that any trigonometrical ratio of an angle is equal to the corresponding co-ratio of its complement.

Find the number of degrees &c. in the angle whose circular measure is 2.

8. Show that $\cos(90^\circ + A) = -\sin A$ when A is an obtuse angle.

II.

1. Explain the convention as to the signs of angles.

Through how many degrees has the hour hand of a clock turned at 20 minutes past 3 p.m., counting from the preceding noon?

2. If an angle contains d degrees and θ radians, prove that

$$\frac{d}{90}=\frac{2\theta}{\pi}.$$

3. Show that $\tan A=\dfrac{\sin A}{\cos A}.$

Prove that (i) $1-\sin^2 A=\dfrac{1}{\sec^2 A}$;

(ii) $(\tan A+\cot A)\sin A\cos A=1.$

4. Trace the changes in $\sin A+\cos A$, as A changes from 0° to 90°.

5. If $\sin A=\frac{1}{3}$ find $\cos A$ and $\cot A$.

6. Prove that $\sin(A+B)=\sin A\cos B+\cos A\sin B$.

Having given that $\sin A=\frac{1}{2}$, $\cos B=\frac{1}{3}$, find $\sin(A+B)$.

7. Show that $\sin 3A+\sin 5A=2\sin 4A\cos A$. Express as a product $\sin 15°+\cos 35°$.

8. Prove the following:

(i) $\cos 2A-\cos 2B=2(\cos^2 A-\cos^2 B)$.

(ii) $\cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$.

(iii) $\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}=\sqrt{2}\sin 15°$.

III.

1. Prove that the circumference of a circle varies as its radius.

Find the area of a circle whose radius is 10 feet.

2. How many radians are there in the following angles: a right angle, the angle of a regular pentagon, of a regular hexagon?

3. If the unit of circular measurement is $2\frac{1}{2}°$, what is the measure of $22\frac{1}{2}°$?

4. Prove the relation $\sec^2 A = \tan^2 A + 1$; show also that

$$(1 - \cot A)^2 + (1 - \tan A)^2 = (\sec A - \operatorname{cosec} A)^2.$$

5. Trace the changes in $\tan A - \cot A$ as A increases from 0° to 90°.

6. What are the *complement* and the *supplement* of an angle? Give the complements of 30°, $\frac{\pi}{6}$, 40^g; and the supplements of 120°, $\frac{2\pi}{3}$, 150^g.

7. Prove that $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$; also that

$$\frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin(A+B)}{\sin(A-B)}.$$

8. If $\sin\alpha = \frac{1}{4}$ and $\cos\beta = \frac{5}{6}$, find the value of $\cos(\alpha+\beta)$.

IV.

1. What is the circular measure of 60°? If the measure of an angle of 10° is 6, what angle is taken for unit?

2. Show that the angle subtended by an arc equal to the radius is the same for all circles.

3. Trace the changes in the sine of an angle as the angle increases from 0° to 180°; also trace the changes in $\sin A + \cos A$ as A increases from 90° to 180°.

4. In the triangle ABC, which has a right angle at C, find a and $\cos B$, given that $c = 13$, $b = 11$.

5. Two persons whose distance apart is 100 yards are situated so that one of them is immediately underneath a balloon, and the angular elevation of the balloon to the other person is 60°. Find the balloon's height.

6. Show that $\sin(180° - A) = \sin A$,

$\cos(180° - A) = -\cos A$.

Deduce that $\sin(360° - B) = -\sin B$.

7. Prove that

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{A-B}{2}.$$

Express as a sum, or difference, the following:

$$\sin A \sin B,\ \tfrac{1}{2}\cos 15°,\ \sin(\alpha+\beta)\cos(\alpha-\beta).$$

8. Show that

$$\sin 2A = 2 \sin A \cos A,\ \cos 2A = \cos^2 A - \sin^2 A.$$

Find the value of $\cos 22\tfrac{1}{2}°$.

V.

1. Show the truth of the equations:

$$\frac{d}{180} = \frac{g}{200} = \frac{\theta}{\pi},$$

where d, g and θ are respectively the number of degrees, grades and radians in an angle.

Find the number of degrees in the angle whose circular measure is $1\tfrac{2}{3}$.

2. If AOP, AOP' are two acute angles such that

$$\sin AOP = \tfrac{1}{4},\ \cos AOP' = \frac{\sqrt{15}}{4},$$

show that O, P and P' lie in one straight line.

3. Give the values of sin 60°, cos 30°, cos 90°, and show that

$$\sin 135° = \frac{1}{\sqrt{2}},\quad \cos 135° = -\frac{1}{\sqrt{2}}.$$

4. Find the value of sin 1170°, tan 2385°, and express cos 2110°, sin (−160°), cot 570°, in terms of trigonometrical ratios of angles of the first quadrant.

5. Prove that $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$.

If $\tan A=\frac{1}{2}$, $\tan(A+B)=\frac{3}{4}$, find $\tan B$.

6. Prove geometrically that $\cos 2A=\cos^2 A-\sin^2 A$; and if $\cos 2A=\frac{1}{4}$, show that $\cos A=\sqrt{5}\div 2\sqrt{2}$.

7. If A, B and C are the angles of a triangle, show that

$$\sin\frac{B+C}{2}=\cos\frac{A}{2}.$$

8. In any triangle, prove that

$$b\sin C=c\sin B,$$

$$c(\sin 2A+\sin 2B)=2\sin C(a\cos A+b\cos B).$$

VI.

1. Express the other trigonometrical ratios in terms of the cosecant.

2. Prove that any trigonometrical ratio of an angle is equal to the corresponding co-ratio of its complement.

3. Show that $\sin(180°-A)=\sin A$, $\cos(180°-A)=-\cos A$; and find the values of $\cos(810°-A)$, $\tan(990°+A)$, $\sin 1260°$.

4. Prove that

(i) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$.

(ii) $\tan A-\tan B=\sin(A-B)\sec A\sec B$.

(iii) $\sin(A+B)\sec B-\cos(A-B)\operatorname{cosec} B$
$$=\cos A(\tan B-\cot B).$$

5. In any triangle show that

$$c=a\cos B+b\cos A.$$

Prove that $$(a+c)\sin\frac{B}{2}=b\cos\frac{A-C}{2}.$$

6. Find the values of $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ in terms of the sides of the triangle.

Show that $\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1.$

7. Prove the formula $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$

VII.

1. Why are *minutes* and *seconds* introduced? How many degrees does the hour hand of a clock turn through in half-an-hour?

2. Find the number of grades in the supplement of the angle of a regular decagon.

3. Prove that the difference of the squares of the cosecant and cotangent of any angle is a constant quantity.

4. Find the values of the cosine and tangent of 60°, 135°, 150°.

The sine of a certain angle is $\frac{3}{8}$; find all the other trigonometrical ratios.

5. Prove the formulæ:

(i) $\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}.$

(ii) $\frac{\sin 3A-\sin A}{\cos 3A-\cos A}=-\cot 2A.$

(iii) $\tan\frac{A}{2}=\frac{\sin A}{1+\cos A}.$

6. Show that in any triangle

$$a^2=b^2+c^2-2bc\cos A.$$

7. Define a logarithm, and find the value of the expression

$$\log_3 81+\log_5 625-\log_6 1296-\log_7 343.$$

8. Having given two angles and a side, show how to solve the triangle. Find the longest side of the triangle ABC, where $AB=1000$ yards, $CAB=35° 10'$, $CBA=83° 15'$, having given that

$$L\cos 28° 25'=9{\cdot}9442409,$$
$$L\cos 6° \ 45'=9{\cdot}9969792,$$
$$\log 11291=4{\cdot}0527383.$$

VIII.

1. An arc of 12 feet subtends at the centre of a circle an angle of 50°; find the radius of the circle, π being taken as equal to 3·1416.

2. If $\sin x=\frac{3}{5}$, $\cos y=\frac{8}{17}$, find $\sin(x+y)$.

3. Prove that

$$\sin^2 A-\cos^2 A\cos 2B=\sin^2 B-\cos^2 B\cos 2A.$$

4. In any triangle show that

$$\frac{\cos 2A}{a^2}-\frac{\cos 2B}{b^2}=\frac{1}{a^2}-\frac{1}{b^2}.$$

5. Find the area of the triangle whose sides are 193, 194 and 195 feet.

6. Write down the general value of the angle $\sec^{-1}(-2)$.

7. If $\cot A+\operatorname{cosec} A=5$, find $\cos A$.

8. If $a=123$, $b=222$, $c=321$, find B, having given that

$$\log 2=\cdot 3010300, \qquad \log 111=2{\cdot}0453230.$$
$$\log 21=1{\cdot}3222193,$$
$$L\tan 14° 38'=9{\cdot}4168099, \text{ diff. for } 1''=86.$$

IX.

1. Define the circular measure of an angle, and find the number of degrees in the unit of circular measure (taking $\frac{22}{7}$ as the approximate value of π).

2. If $\tan\theta = \dfrac{b}{\sqrt{a^2-b^2}}$, prove that

$$\sin\theta(1+\tan\theta)+\cos\theta(1+\cot\theta)-\sec\theta=\frac{a}{b}.$$

3. Prove the formulæ:

(α) $\cos(A-B)=\cos A\cos B+\sin A\sin B$.

(β) $2\cos\dfrac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}$.

(γ) $\tan\left(45^\circ-\dfrac{\theta}{2}\right)=\dfrac{\cot\theta}{1+\operatorname{cosec}\theta}$.

4. If A lies between 180° and 270°, and the numerical value of its sine is $\frac{1}{5}$, determine $\cos\dfrac{A}{2}$ by means of the expression (β) of the last question.

5. Show that in any triangle the sides are proportional to the sines of the opposite angles.

6. If a straight line be drawn bisecting the angle A of a triangle ABC to meet the opposite side in D; show that the segments of this side are

$$a\sin C/(\sin B+\sin C), \text{ and } a\sin B/(\sin B+\sin C).$$

7. If in a triangle ABC, $A=60^\circ$, $a=\sqrt{3}$, $b=\sqrt{2}$, solve the triangle and show that its area is $\dfrac{3+\sqrt{3}}{4}$.

8. Show that the radius of the inscribed circle is $\dfrac{S}{s}$. In a right-angled triangle whose right angle is C, prove that $r+c=s$.

X.

1. Prove that

$$(1-\cos\theta)\{\sec\theta+\operatorname{cosec}\theta(1+\sec\theta)\}^2=2\sec^2\theta(1+\sin\theta).$$

2. A man wishes to measure the distance between two points A and B, between which lies an obstacle. He therefore walks from A to C in a direction at right angles to AB a distance of

50 yards. He now finds that he can walk directly from C to B, and that CB makes an angle of 60° with AC. Find the distance from A to B.

3. Prove that

$$\cos(\alpha-\beta)\cos 2\beta - \sin(\alpha-\beta)\sin 2\beta$$
$$= \cos(\beta-\alpha)\cos 2\alpha - \sin(\beta-\alpha)\sin 2\alpha.$$

If $\cos x = \frac{4}{5}$, $\sin y = \frac{8}{17}$, find $\cos(x+y)$.

4. Show that $\cos 4A = \cos^4 A + \sin^4 A - 6\sin^2 A\cos^2 A$.

5. If ABC be an equilateral triangle each of whose sides is 16 inches long, and in AB a point P be taken 10 inches from A, show that CP is 14 inches.

6. A base-line 300 feet in length is measured from the foot of a vertical tower, and at the end of this line the angular elevation of the top of the tower is observed to be 33° 41′ 24″. Show that the height of the tower is very nearly 200 feet. Having given that $L\tan 33° 41' = 9{\cdot}8237981$, diff. for $1' = 2738$,

$$\log 3 = {\cdot}4771213.$$

7. Solve completely the equation $\cos^2\theta = \frac{1}{2}$.

8. A line is drawn through the vertex A of a triangle ABC dividing it into two triangles. Show that the ratio of the radii of their circum-circles is equal to b/c.

XI.

1. Does the value of $\sec\theta$ got from the equation

$$\sec^2\theta = \frac{1-2\cos^2\alpha}{1-\cos^2\alpha}$$

give a possible value for θ?

2. Prove that

$$\sin(-A) = -\sin A,$$
$$\tan(90°+A) = -\cot A.$$

Given $\cos\theta = \frac{20}{101}$, find $\cot(90°+\theta)$.

3. If $2A+B = 90°$, then

$$\cos A = \sqrt{\tfrac{1}{2}(1+\sin B)}.$$

4. Prove the formulæ:

(i) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B.$

(ii) $1+\cos 4A=2\cos 2A\,(1-2\sin^2 A).$

5. Find θ from the equation

$$\sin\theta+\sin 2\theta+\sin 3\theta+\sin 4\theta=0.$$

6. If F is the middle point of the side AB of a triangle, and N is the foot of the perpendicular from C on AB, then

$$\frac{\sin(A-B)}{2NF}=\frac{\sin A}{a}.$$

7. C is the centre of the circle inscribed in a sector of a circle whose angle is 60°. From C the lines CD and CF are drawn at right angles to the bounding radii of the sector. Find the ratio of the area of the given sector to that of the smaller sector thus formed in the inscribed circle.

XII.

1. Define the sine and cosine, and prove that,

$$\sec^2 A=1+\tan^2 A.$$

Find the relation between $\sin A$ and $\tan A$.

In the triangle ABC, A is joined to D the middle point of BC, show that $\quad \operatorname{cosec} BAD : \operatorname{cosec} CAD :: AB : AC.$

2. Prove that $\tan(180+A)=\tan A$, and express in terms of positive angles not greater than a right angle,

$$\sin 3485°,\ \operatorname{cosec}(-3970°).$$

3. Write down the smallest positive and negative values of θ which satisfy the equations

(i) $\sqrt{2}\sin\theta=-1.$

(ii) $\tan\theta=-1.$

4. If A and B are positive angles whose sum is less than a right angle, show that

$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

Also show that,

(α) $\tan 3A = (3\tan A - \tan^3 A) \div (1 - 3\tan^2 A)$.

(β) $$\sin^2 \frac{A+B}{2} \cos^2 \frac{A-B}{2} + \cos^2 \frac{A+B}{2} \sin^2 \frac{A-B}{2} = 1 - \tfrac{1}{2}\cos^2 A - \tfrac{1}{2}\cos^2 B.$$

5. Solve the equations:

(i) $\cot A - \operatorname{cosec} 2A = 1$.

(ii) $\cos^3 A \sin 3A + \sin^3 A \cos 3A = \dfrac{3\sqrt{3}}{8}$.

6. Find $\sin 18°$, and by means of (α) in question 4, find $\tan 15°$.

Prove the following:

(i) $\sin A = \sin 3A + \sin 30°$, where $A = 54°$.

(ii) $\sin^{-1}\frac{5}{13} + \tan^{-1}\frac{7}{24} = \cos^{-1}\frac{253}{325}$.

7. What are the formulæ for solving a triangle when the three sides are known? Prove that

$$\sin\frac{A-B}{2} = \frac{a-b}{c}\cos\frac{C}{2}.$$

8. If S is the area of the triangle ABC, and S' that of its pedal triangle, show that

$$\frac{S'}{S} = 2\frac{a'b'c'}{abc},$$

where a', b', c' are the sides of the pedal triangle.

XIII.

1. Express the cosine and the tangent of an angle in terms of each other. Investigate a formula for all angles which have a given cosine.

If $\sin A \cdot \tan A = 1$, show that $\cos A = \dfrac{\sqrt{5}-1}{2}$.

2. Find the sine, cosine, and tangent of 15° and 18°.

3. Investigate the value of $\tan(A+B)$ in terms of $\tan A$ and $\tan B$, and thence deduce the value of $\tan(A+B+C)$ in terms of $\tan A$, $\tan B$ and $\tan C$.

If $\tan 3A+\tan 2A=0$, show that $\tan A$ may have any of the following values,

$$0,\ \pm\sqrt{5\pm 2\sqrt{5}}.$$

4. Prove that $2\cos\frac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}$; explain the origin of the double ambiguities in the expressions.

5. Assuming that $\sin 210^\circ=-\frac{1}{2}$, find the values of $\cos 105^\circ$ and $\sin 105^\circ$.

$$\left[-\frac{\sqrt{3}-1}{2\sqrt{2}},\ \frac{\sqrt{3}+1}{2\sqrt{2}}\right].$$

6. Prove that $\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$.

If $\sin 3A=\sin 2A$, show that A may have any value included in $2n\pi$, or $\frac{2n+1}{5}\pi$, got by giving any integral value to n.

7. Prove that

$$2\cos\frac{A}{2^n}=\sqrt{2+\sqrt{2+\sqrt{\ldots\ldots\ldots\sqrt{2+2\cos A}}}};$$

the symbol indicating the extraction of the square root being repeated n times.

8. Trace the changes in sign of $\frac{\sin(\pi\cos\theta)}{\cos(\pi\sin\theta)}$, as θ increases from 0 to π.

XIV.

1. Define the various trigonometrical ratios and express all in terms of the tangent.

2. Prove the formula, $\sin A=\cos(90^\circ-A)$, for the case where A is an obtuse angle.

Show that if an angle lie between $4\frac{1}{2}$ and 5 right angles, its cosine is less than its sine, and if between $6\frac{1}{2}$ and 7 right angles greater.

3. Prove geometrically that

$$\tan A = \frac{\sin 2A}{1+\cos 2A}.$$

Show that $\cos(A+B+C)$

$$= 4\cos A\cos B\cos C - \cos A\cos(B-C) - \cos B\cos(C-A) - \cos C\cos(A-B).$$

4. Prove by means of one figure that, A being an obtuse angle,

$$\sin(90+A) = \cos A, \quad \sin(A-90^\circ) = -\cos A,$$

$$\cos(90+A) = -\sin A, \quad \cos(A-90) = +\sin A.$$

5. Prove the formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

6. Show that if $\sin A \sin B < (\cos A + \cos B)(1+\cos C)$, C being the least angle of a triangle, then the sum of any two of the perpendiculars from the angles on the opposite sides will be greater than the third.

7. Given the two sides of a triangle and the included angle, show how to find its other parts.

If $b=5$, $c=1$, $A=60^\circ$, find B and C having given that

$$\log 2 = \cdot 30103, \quad L\tan 49^\circ\, 6' = 10\cdot 06237$$

$$\log 3 = \cdot 47712, \quad L\tan 49^\circ\, 7' = 10\cdot 06262.$$

8. What is the "ambiguous case" in the solution of triangles. Discuss the following cases,

(α) $a=10$, $b=21$, $A=30^\circ$;

(β) $a=2$, $b=\sqrt{3}$, $A=45^\circ$;

(γ) $a=2\sqrt{3}$, $b=4$, $A=60^\circ$;

(δ) $a=3$, $b=4$, $A=60^\circ$.

XV.

1. Show that the circumference of a circle varies as its radius.

The four angles of a quadrilateral are in A.P., and the difference of the greatest and least is equal to a right angle. Express each of the four angles in degrees and also in circular measure.

2. Define the cosecant and cotangent of an angle and prove that $\operatorname{cosec}^2 A + \operatorname{cosec}^2 (90° - A) = \operatorname{cosec}^2 A \,.\, \operatorname{cosec}^2 (90 - A)$.

If $\sin\theta = \frac{45}{53}$, find $\cos\theta$ and $\tan\theta$.

3. Prove that

$$\cos B - \cos A = 2 \sin \tfrac{1}{2}(A - B) \sin \tfrac{1}{2}(A + B).$$

Show also that

$$\cos (A + B) \cos (A - B) + 1 = \cos^2 A + \cos^2 B,$$

$$\tan 5A \tan 3A \tan 2A = \tan 5A - \tan 3A - \tan 2A.$$

4. From the equation $\sin 3A = 3 \sin A - 4 \sin^3 A$, find $\sin A$, given that $\sin 3A = \frac{23}{27}$.

5. Prove that in any triangle

$$a \cos B + b \cos A = c.$$

Show also that

$$\frac{1}{a}\cos^2\frac{A}{2} + \frac{1}{b}\cos^2\frac{B}{2} + \frac{1}{c}\cos^2\frac{C}{2} = \frac{s^2}{abc}.$$

6. Find the formula for solving a triangle when two sides and the included angle are given. Show how the triangle may be solved if the difference of two angles and the sides opposite to them are given.

7. Find the radius of the circumscribing circle of a triangle.

If O is the centre of this circle in the triangle ABC, show that the areas BOC, COA, AOB are proportional to $\sin 2A$, $\sin 2B$, $\sin 2C$.

XVI.

1. Show that the measure of the length of an arc of a circle is equal to the product of the measure of the radius and the circular measure of the angle subtended by the arc at the centre.

Given that $\pi = 3{\cdot}14159$, find the length of an arc of a circle which subtends at the centre an angle of $15°$, the radius of the circle being 12 feet.

2. Show that $\sin(n\pi + (-1)^n \alpha) = \sin \alpha$.

Write down the general value of θ when $\sin^2\theta = \sin^2\alpha$.

3. Prove geometrically, for the case when A is acute,

$$\cos 2A = \cos^2 A - \sin^2 A, \quad \sin 2A = 2\sin A \cos A.$$

4. Prove the following identities:

(1) $\tan(A+B)\tan(A-B) = (\tan^2 A - \tan^2 B) \div (1 - \tan^2 A \tan^2 B)$.

(2) $\sin 3A + \cos 3A = (\cos A - \sin A)(1 + 4\sin A \cos A)$.

(3) $\dfrac{\sin\theta + \sin 5\theta}{\sin 2\theta + \sin 4\theta} = \cos 2\theta \sec\theta$.

5. In any triangle show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Show also that

$$(b^2 - c^2)\cot A + (c^2 - a^2)\cot B + (a^2 - b^2)\cot C = 0.$$

6. Show that the radius of the inscribed circle of a triangle is

$$a \sin\frac{B}{2}\sin\frac{C}{2}\sec\frac{A}{2}.$$

Prove also that
$$\frac{1}{r_2} - \frac{1}{r_3} = \frac{c-b}{rs}.$$

7. From the top of a tower a person observes the depressions, α and β, of two distant points in the horizontal plane at the foot of the tower, and also the angle θ subtended at his eye by the line joining the points. The distance of these two points from

each other he knows to be a miles. Shew that the height of the tower is

$$\frac{a}{(\operatorname{cosec}\alpha+\operatorname{cosec}\beta)\cos\phi},$$

where

$$\sin\phi=\frac{2\sqrt{\operatorname{cosec}\alpha\operatorname{cosec}\beta}\cos\frac{\ }{2}}{\operatorname{cosec}\alpha+\operatorname{cosec}\beta}.$$

8. Define $\sin^{-1}x$, and show that

$$\{\cos(\sin^{-1}x)\}^2=\{\sin(\cos^{-1}x)\}^2.$$

XVII.

1. Trace the changes in sign and magnitude of $\sin(\pi\cos\theta)$ and of $\sin\theta+\sin 2\theta$, as θ increases from 0° to 180°.

2. Find the most general solution of the equation

$$8\sin^2\theta=3-\sqrt{5}.$$

3. Show that if the value of $\sin A$ be known, then $\sin\frac{A}{2}$ may be any one of 4 numbers.

Having given that $\sin 143°=\cdot6018150$,

show that $2\sin 71°\,30'=1\cdot8966$ nearly.

4. Show that the area of a triangle whose sides are a, b, c is equal to $\frac{1}{4}\sqrt{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}$.

5. If in a triangle $b=520$, $c=460$, $A=62°\,36'$ find B and C, having given

$$\log 3=\cdot4771213,\ \log 7=\cdot8450980,$$
$$L\cot 31°\,18'=10\cdot2160896,$$
$$L\tan 5°\,45'=9\cdot0030066,\ L\tan 5°\,46'=9\cdot0042721.$$

6. The elevation of a tower from a point A due N. of it is observed to be 45°, and from a point B due E. of it to be 30°. If $AB=240$ feet, find the height of the tower.

7. If m_1 and m_2 are the lines joining the vertices A and B of the triangle ABC to the middle points of the opposite sides, show that $$4(m_1^2 - m_2^2) = 3(b^2 - a^2).$$

8. Show that the area of a quadrilateral inscribed in a circle is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$. If this quadrilateral is also capable of having a circle inscribed in it, show that its area is $\sqrt{abcd}$.

XVIII.

1. Find the circular measure of the angle $29° 10'$ (taking $\frac{22}{7}$ as the approximate value of π).

2. Prove that
$$(1+\sin A)^2 \{\cot A + 2 \sec A (1 - \operatorname{cosec} A)\} + \operatorname{cosec} A \cos^3 A = 0.$$

3. Find all the positive values of θ between 0 and 2π which satisfy the equation $\sec^2\theta \operatorname{cosec}^2\theta + 2\operatorname{cosec}^2\theta = 8$.

4. Prove geometrically that
$$\cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$
Also show that
$$\cos(B+C)\cos(B-C) - \cos(A+C)\cos(A-C) = \sin(A+B)\sin(A-B).$$

5. Find all the angles of the triangle ABC in which $a = 6{\cdot}1$ inches, $b = 4{\cdot}5$ inches and $B = 38°$, having given
$$\log 61 = 1{\cdot}78533, \quad L \sin 56° 34' = 9{\cdot}92144,$$
$$\log 45 = 1{\cdot}65321, \quad L \sin 56° 35' = 9{\cdot}92152,$$
$$L \sin 38° = 9{\cdot}78934.$$

6. If r_1 be the radius of the circle escribed to the triangle ABC opposite the angle A, and R the radius of the circum-circle, prove that $$r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

The perimeter of the triangle formed by the feet of the perpendiculars from the angles on the opposite side is
$$4R \sin A \sin B \sin C.$$

7. Find all the solutions of the equation

$$2\sin^2 x = \cos^2\frac{3x}{2}.$$

8. If the diagonals of a quadrilateral $ABCD$ intersect in O, show that

$$\text{area } AOB \,.\, \text{area } ABCD = \text{area } ABC \,.\, \text{area } ABD.$$

XIX.

1. Define the sine, cosine, and tangent of an angle and show how the definition includes angles of any magnitude.

Trace the changes in sign and magnitude of $\tan\theta + \sec\theta$, as θ changes from $0°$ to $360°$.

2. If $A+B$ be less than $90°$, find $\cos(A+B)$ in terms of the sine and cosine of A and B.

Prove that

(i) $\cos(15° - a)\sec 15° - \sin(15° - a)\operatorname{cosec} 15° = 4\sin a.$

(ii) $$\frac{\cot A + \cot B}{\tan A + \tan B} + \frac{\cot B + \cot C}{\tan B + \tan C} + \frac{\cot C + \cot A}{\tan C + \tan A} = 1,$$

where $$A + B + C = 180°.$$

3. Find expressions for all the values of θ which satisfy the equation $$\sin 4\theta = \sin\theta.$$

If θ_1 and θ_2 be values of θ which satisfy the equation

$$a\cos\theta + b\sin\theta = c,$$

then $$\frac{\cos\frac{\theta_1+\theta_2}{2}}{a} = \frac{\sin\frac{\theta_1+\theta_2}{2}}{b} = \frac{\cos\frac{\theta_1-\theta_2}{2}}{c}.$$

4. Prove that in any triangle

$$\frac{a}{\sin A} = \frac{a^2 - b^2}{c\sin(A-B)}.$$

5. If I be the centre of the inscribed circle, and P the orthocentre, then

$$IP^2 = R^2 (1 - 8 \cos A \cos B \cos C).$$

6. Two objects P, Q are observed from stations A, B. It is found that AP, AQ make angles α, β, respectively with the line BA produced; BP, BQ make angles α', β' with the same line; prove that the area of the triangle PAQ is

$$AB^2 \frac{\sin \alpha' \sin \beta' \sin (\beta - \alpha)}{2 \sin (\alpha - \alpha') \sin (\beta - \beta')}.$$

7. Show that the distances from A, one of the angular points of a regular octagon $ABCDEFGH$, from B, C, D respectively are in the ratio

$$\sqrt{2-\sqrt{2}} : \sqrt{2} : \sqrt{2+\sqrt{2}}.$$

8. Prove that

$$\sin A (\sin 2A + \sin 4A + \sin 6A) = \sin 3A \sin 4A.$$

9. In any triangle show that

$$(b^2 - c^2) \cot^2 \frac{A}{2} + (c^2 - a^2) \cot^2 \frac{B}{2} + (a^2 - b^2) \cot^2 \frac{C}{2}$$
$$= -\frac{1}{r^2} (a+b+c)(b-c)(c-a)(a-b).$$

10. From the equations,

$$\cos x + \cos y + \cos \alpha = 0,$$
$$\sin x + \sin y + \sin \alpha = 0;$$

deduce the equations,

$$\cot \frac{x+y}{2} = \cot \alpha, \quad \cos \frac{x-y}{2} = \pm \frac{1}{2}.$$

XX.

1. If A and B be each between 0° and 90°, prove geometrically that

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B).$$

Prove the formulæ:

(a) $\cot A - \operatorname{cosec} 2A = \cot 2A$.

(β) $$\frac{\sin(3a+5\beta)+\sin(5a+3\beta)}{\sin 2a+\sin 2\beta} = 4\cos(a+\beta)\cos(2a+2\beta).$$

2. Given that $\tan\theta = \frac{3}{4}$, find the values of $\sin\theta$ and $\tan 2\theta$, and explain why it is that $\sin\theta$ has two values and $\tan 2\theta$ only one value.

3. Find an expression for all angles which have the same cosine. If $2\cos(3x-45°)=1$, show that $x = n\,.\,120° \pm 20° + 15°$, where n is any positive or negative integer.

4. Find the cosine of an angle in terms of the sides, and prove that in the triangle ABC

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Employ the formula to find the angle A, having given

$$a=29,\quad b=18,\quad c=23.$$

$$\log 34 = 1{\cdot}5314789,\quad L\tan 44°\,35' = 9{\cdot}9936832,$$
$$\log 35 = 1{\cdot}5440680,\quad L\tan 44°\,36' = 9{\cdot}9939359.$$

5. A spectator ascends a tower to a small window at a height h, and observes the angles of depression of two objects A and B on the horizontal plane on which the tower stands and in the same vertical plane with the window to be 30° and 15°.

Show that $$AB = 2h.$$

Prove also that if the spectator ascends a further height $2h(\sqrt{3}+1)$, the distance AB will again subtend the angle 15° at his eye.

6. Find the magnitude of the radius of the inscribed circle of a triangle ABC; if I is the centre of the inscribed circle show that the radius of the circle inscribed in the triangle BIC is

$$\sqrt{2}a\,\frac{\sin\frac{B}{4}\sin\frac{C}{4}}{\cos\frac{A}{4} - \sin\frac{A}{4}}.$$

7. Show that the area of a quadrilateral $ABCD$ inscribed in a circle is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$.
If $a=c$ prove that $\cos B=(b-d)/2a$.

8. How does the solution of a trigonometrical equation differ from that of an algebraic one?

Solve the equations,

(i) $\cos\theta-\sqrt{3}\sin\theta=1$.

(ii) $\cos\theta+\cos\alpha=\sin\theta+\sin\alpha$.

ANSWERS TO EXAMPLES.

I. (Page 8.)

1. (1) 50^g. (2) 20^g. (3) 30^g. (4) 110^g.
(5) $3^g\,67'\,50''$. (6) $59^g\,16'\,94''$. (7) $120^g\,37'\,50''$.
(8) $13^g\,65'\,15{\cdot}4''$. (9) $13'\,33''{\cdot}\dot{3}$. (10) $5^g\,72'\,65''{\cdot}4$.
(11) $192^g\,51'\,97''$. (12) $30^g\,19'\,1{\cdot}2''$.

2. (1) $22°\,30'$. (2) $9°$. (3) $13°\,30'$. (4) $54'$.
(5) $11°\,51'\,51''{\cdot}3$. (6) $112°\,35'\,30''{\cdot}12$. (7) $31°\,57'$.
(8) $3''{\cdot}24$. (9) $9°\,8'\,29''{\cdot}652$. (10) $62°\,14'\,8''{\cdot}592$.
(11) $54'\,32''{\cdot}724$. (12) $10'\,51''{\cdot}888$.

3. $19°$, $1°$. **4.** $36°$, $18°$. **5.** $36°$, $72°$, $72°$.

6. $30°$. **9.** $\cdot 22$ nearly.

II. (Page 13.)

1. (1) $60°$. (2) $18°$. (3) $15°$. (4) $3°\,20'$.
(5) $270°$. (6) $240°$. (7) $57°\,17'\,44''{\cdot}81$.
(8) $572°\,57'\,28''{\cdot}1$. (9) $1°\,8'\,45''$ nearly. (10) $85°\,56'\,36''$.
(11) $34°\,21'\,49''$. (12) $18°\,13'$ nearly.

2. (1) $\frac{\pi}{4}$. (2) $\frac{\pi}{3}$. (3) $\frac{5\pi}{6}$. (4) $\frac{\pi}{4}$. (5) $\frac{\pi}{36}$.
(6) $\frac{\pi}{180}$. (7) $\frac{\pi}{720}$. (8) $\frac{3\pi}{4000}$. (9) 10π.
(10) $\cdot 08575\pi$. (11) $\cdot 018275\pi$. (12) $\cdot 12847\dot{2}\pi$.
(13) $\cdot 335943\pi$. (14) $\cdot 46827\pi$. (15) $\frac{\pi^2}{180}$.

III. (Page 15.)

1. 628 yards nearly. 2. 4000 miles. 3. 49 feet. 4. $\frac{1}{22}$.
5. 5 yds. 7. $7\frac{7}{11}$ inches nearly. 8. 6°. 9. $15° 12' 45''$.

IV. (Page 17.)

1. $\frac{19\pi}{3}$, 1140°. 3. 19·8 feet nearly. 4. $\frac{\pi}{3}$.
5. $314\frac{2}{7}$ sq. feet. 6. 9π sq. inches $= 28\frac{2}{7}$ sq. inches nearly.
7. $\frac{\pi r^2}{3}, \frac{\pi r^2}{6}$. 8. $\frac{\sqrt{\pi}}{3}$. 9. 26·58 feet.
10. ·39 of a mile nearly. 11. 1·309 feet. 12. $\frac{6}{7}$.
13. 100°, 80°. 14. 1·087, ·913. 16. 2600^g.
17. 30°, 60°, 90°; $33\frac{1}{3}^g$, $66\frac{2}{3}^g$, 100^g. 18. 40°, 60°, 80°.
19. 39°, 60°, 81°. 20. $64\frac{2}{7}$°.
21. (i) 108°, 120^g; (ii) 120°, $133·3^g$; (iii) 144°, 160^g.
22. 6°. 23. 128°. 24. $\frac{3}{4}, \frac{3}{2\pi}$. 25. 6.
26. ·50625. 27. $\frac{1}{3}$. 28. $59° 59' 58''$, $\frac{25\pi}{36}$.
29. 1.1″ nearly. 30. 2094·4 miles approx.
31. $65° 24' 30''·38$. 32. $a=89$, $b=30$.

VI. (Page 27.)

1. $\cos A = \frac{\sqrt{3}}{2}$, $\tan A = \frac{1}{\sqrt{3}}$, $\operatorname{cosec} A = 2$, $\sec A = \frac{2}{\sqrt{3}}$, $\cot A = \sqrt{3}$.
2. $\sin A = \frac{1}{\sqrt{2}}$, $\tan A = 1$, $\operatorname{cosec} A = \sec A = \sqrt{2}$, $\cot A = 1$.
3. $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$, $\operatorname{cosec} A = \frac{5}{4}$, $\sec A = \frac{5}{3}$, $\cot A = \frac{3}{4}$.
4. $\sin A = \frac{1}{4}$, $\cos A = \frac{\sqrt{15}}{4}$, $\tan A = \frac{1}{\sqrt{15}}$, $\sec A = \frac{4}{\sqrt{15}}$, $\cot A = \sqrt{15}$.
5. $\cos A = \frac{11}{61}$, $\sin A = \frac{60}{61}$, $\tan A = \frac{60}{11}$, $\cot A = \frac{11}{60}$, $\operatorname{cosec} A = \frac{61}{60}$.
6. $\sin A = \frac{\sqrt{6} - \sqrt{2}}{4}$, $\cos A = \frac{\sqrt{6} + \sqrt{2}}{4}$, $\tan A = 2 - \sqrt{3}$,
$\sec A = \sqrt{6} - \sqrt{2}$, $\operatorname{cosec} A = \sqrt{6} + \sqrt{2}$.

IX. (PAGE 41.)

1. $\frac{1}{\sqrt{2}}-\frac{1}{2}$. 2. $\frac{4}{3}$. 3. $\sqrt{\frac{3}{2}}$. 4. 3. 5. 5.
6. 0. 7. 1. 8. $\frac{4}{\sqrt{3}}-\frac{1}{\sqrt{2}}-\frac{3}{4}$.
9. $\frac{11}{2}-\frac{3\sqrt{3}}{2}$. 10. $\frac{9}{4}$. 11. $4+\frac{2}{\sqrt{3}}$.

X. (PAGE 44.)

1. $A=60°$, $B=30°$, $c=2\sqrt{21}$. 2. $A=45°$, $B=45°$, $c=4\sqrt{2}$.
3. $A=30°$, $B=60°$, $b=6\sqrt{3}$. 4. $A=60°$, $B=30°$, $b=2$.
5. $A=45°$, $B=45°$, $b=\sqrt{2}$. 6. $B=60°$, $a=5$, $b=5\sqrt{3}$.
7. $B=75°$, $a=3(\sqrt{6}-\sqrt{2})$, $b=3(\sqrt{6}+\sqrt{2})$.
8. $B=30°$, $c=10$, $b=5$. 9. $B=45°$, $b=10$, $c=10\sqrt{2}$.
10. $A=14°\,28'\,21''$, $B=75°\,31'\,39''$, $b=11{\cdot}6196$.

XI. (PAGE 49.)

1. 115 feet 4 inches nearly. 2. $57\frac{3}{5}$ feet. 3. 30°.
4. 28·8 yards nearly. 5. 19 nearly. 6. 155·7 yards.
7. 7 feet. 8. 73·2 feet. 9. $75(1+\sqrt{3})$ feet.
10. 75 feet. 11. 236 feet, 136 feet. 12. 346·4 feet.
13. $25(\sqrt{3}+1)$ feet. 14. $AB=86{\cdot}6$, $BC=50$. 15. 30°.
17. $45\frac{1}{4}$ yards. 18. $\frac{\sqrt{3}}{2}$ of a mile. 19. 250 yards.
20. $\frac{\sqrt{3}}{2}$ of a mile. 21. 86·6 feet. 22. 72 feet.
23. 200 feet. 26. one inch.
27. $18(2+\sqrt{3})$ feet, $6(2+\sqrt{3})$ feet.
28. length $=\sqrt{2b}$, breadth $=2b$. 29. $125(\sqrt{3}+3)$ feet.

XII. (PAGE 72.)

1. (2) $-\frac{1}{2}$, (3) $-\sqrt{3}$, (4) $-\sqrt{3}$, (5) $-\sqrt{2}$, (6) $-\frac{\sqrt{3}}{2}$,
(7) $\frac{1}{\sqrt{3}}$, (8) $-\frac{1}{\sqrt{2}}$.

2. (1) $-\sqrt{3}$, (2) -1, (3) -2, (4) $\frac{1}{\sqrt{2}}$, (5) -1.

4. 2. **5.** (1) $-\sin A$, (2) $-\sin A$, (3) $-\cot A$, (4) $-\cot A$, (5) $\sin A$, (6) $\cos A$, (7) $\cot A$, (8) $\frac{2}{\sqrt{3}}$.

XIII. (Page 79.)

11. $\frac{2499}{2501}$. **12.** $\frac{33}{65}$. **13.** $\frac{812}{1037}$.

14. $\frac{16\sqrt{798}+3}{565}$. **15.** $\frac{5}{8\sqrt{2}+3}$.

XIV. (Page 83.)

3. $\frac{23}{25}$, $\frac{4\sqrt{6}}{25}$. **4.** $-\frac{24}{7}$.

XVI. (Page 90.)

1. $\sin 2\alpha + \sin 2\beta$. **2.** $\sin 4\alpha + \sin 2\alpha$. **3.** $\sin 38° - \sin 16°$.
4. $\sin 3\theta + \sin \theta$. **5.** $\cos 12A + \cos 2A$. **6.** $\cos 8\theta - \cos 10\theta$.

XIX. (Page 110.)

4. $-\frac{23}{27}$. **5.** $\frac{11}{16}$.

XXI. (Page 126.)

1. (i) $++$, (ii) $++$, (iii) $--$, (iv) $-+$.
2. (i) $+-$, (ii) $+-$, (iii) $-$, (iv) $-$.
3. (i) $+$, (ii) $+$, (iii) $+$, (iv) $+$.

XXII. (Page 134.)

4. 691 yards nearly. **5.** $\frac{22a}{7 \times 6 \times 180}$. **6.** $94\frac{2}{7}$ yards.

XXVII. (Page 153.)

1. $0, \frac{3}{4}, \frac{7}{4}$. **2.** 2·79588, 3·39794. **3.** $-2, -\frac{3}{2}, -\frac{1}{2}$.

4. 1·8750613, $\bar{4}$·6532126, $\bar{2}$·6930604.

5. 3·2886963, $\bar{3}$·4771213, − 2·4771213.
7. ·4149733. 8. eighteen digits, twenty-one ciphers.
10. $\bar{2}$·77509556. 11. 1·449559 nearly.

XXVIII. (Page 158.)

1.	·5630853.	2.	·8258469.	3.	34° 11′ 39″·8.
4.	·2122724.	5.	·65121948.	6.	·8054429.
7.	1·16759196.	8.	2·88163677.	9.	·2775594.
10.	·97684374.	11.	3·1079046.	12.	·731357.
13.	·9510550.	14.	1·3031731.	15.	·2276941.
16.	30° 0′ 1″·1.	17.	17° 27′ 27″·3.	18.	19° 28′ 16″·3.
19.	25° 51′ 30″·9.	20.	26° 33′ 54″·1.	21.	41° 24′ 34″·6.
22.	82° 49′ 9″·2.	23.	54° 2′ 41″·6.	24.	75° 57′ 49″·5.
25.	78° 41′ 24″.	26.	15° 8′ 32″·1.	27.	46° 14′ 1″·7.

XXIX. (Page 164.)

1.	9·3690612.	2.	9·6687660.	3.	9·6989736.
4.	9·9375294.	5.	9·8495903.	6.	9·6524836.
7.	9·2607238.	8.	10·5770862.	9.	10·6372318.
10.	10·4732944.	11.	10·5034627.	12.	9·7500491.
13.	31° 24′ 22″·1.	14.	8° 49′ 4″·5.	15.	76° 26′ 2″·7.
16.	23° 12′ 25″.	17.	12° 13′ 33″·7.	18.	51° 7′ 5″3.
19.	69° 57′ 15″·9.	20.	30° 52′ 38″.		

XXX. (Page 167.)

1.	13·17647, 21·92728.	2.	3210·793 ft.
3.	1·029729, 1·304883.	4.	13·47296, 15·32089.
5.	15·396.	6.	16·28637, 24·59028.
7.	9·042839, 28·90945.	8.	10·25719, 20·97454.
9.	7·072344, 7·142524.	10.	258·3323, 241·321.

XXXI. (Page 170.)

1. $\frac{C}{2} = 69° 17′ 42″·6$. 2. $\frac{A}{2} = 6° 17′ 20″·6$.

3. Half the greatest angle $= 34° 26′ 21″·226$.

4. $A=51°\,19'\,4''$, $B=18°\,11'\,41''$.
5. $50°\,28'\,43''\cdot6$, $58°\,58'\,32''\cdot6$. 6. $8°\,27'\,40''\cdot4$, $34°\,54'\,16''\cdot6$.
7. $19°\,11'\,17''$, $99°\,35'\,28''\cdot6$. 8. $32°\,57'\,6''$, $51°\,10'\,58''\cdot2$.
9. $53°\,7'\,48''$, $59°\,29'\,23''$, $67°\,22'\,49''$.

XXXII. (PAGE 174.)

1. $116°\,33'\,54''$, $26°\,33'\,54''$. 2. $6°\,1'\,53''\cdot9$, $108°\,58'\,6''\cdot1$.
3. $71°\,44'\,29''\cdot5$, $48°\,15'\,30''\cdot5$. 4. $90°$, $30°$, $2\sqrt{3}$.
5. $88°\,28'\,2''$,, $11°\,31'\,58''$, $c=4\cdot925$.
6. $98°\,6'\,3''$, $56°\,53'\,57''$, $5\cdot54941$.
7. $126°\,2'\,34''\cdot5$, $37°\,57'\,25''\cdot5$, $39\cdot885$.
8. $96°\,6'\,25''\cdot8$, $21°\,53'\,34''\cdot2$, $7\cdot1039$.
9. $102°\,52'\,52''\cdot5$, $37°\,7'\,7''\cdot5$, $13\cdot847$.

XXXIII. (PAGE 180.)

1. $18°\,26'\,5''\cdot8$. 2. $18°\,12'\,35''\cdot9$, $15\cdot1987$.
3. $11\cdot91753$, $15\cdot5572$. 4. $B=90°$. 5. $2c=5\sqrt{5}\pm\sqrt{15}$.
6. $B=90°$. 7. $113°\,30'$, $26°\,30'$.
8. $B=90°$, ambiguous, no triangle.
9. $60°$, $75°$, $\sqrt{6}$; or $30°$, $105°$, $\sqrt{2}$.
10. $35°\,5'\,48''$. 11. $34°\,8'\,15''\cdot2$.
12. $44°\,37'\,42''$, $82°\,50'\,3''$; or $30°\,17'\,48''$, $97°\,9'\,57''$.
13. $c=14$, $P=12$. 14. $70°\,53'\,36''$, $49°\,6'\,24''$.
15. 382094 sq. feet. 16. $144°\,8'\,15''$, $29°\,51'\,45''$, $229\cdot4536$.
17. $4227\cdot481$.
18. Side opposite 70 is $1035\cdot43$, other side $765\cdot4321$.
19. $67°\,58'\,32''\cdot5$, $44°\,2'\,55''\cdot5$. 20. $137°\,14'\,25''$, $4°\,19'\,35''$.
21. Angle opposite 394 is $39°\,35'\,11''$, the other angle is $28°\,20'\,49''$.
22. $66\cdot07845$.

XXXIV. (PAGE 189.)

4. 2879 feet nearly. 5. $331\cdot6912$ feet. 6. $14\cdot9394$ feet.
7. $2032\cdot82$ yards. 8. $\cdot2039$ of a mile. 9. $2118\cdot05$ yards.
11. $155\cdot8$ feet. 12. $212\cdot85$ feet, $618\cdot17$ feet.
14. $291\cdot49$ yards.

XXXV. (PAGE 195.)

3. $DP = 238{\cdot}5$, $DQ = 480{\cdot}5$.
8. 10 miles an hour.
10. 100 feet, 25 feet.
11. 120 feet.
14. $25(\sqrt{10}+\sqrt{2})$ feet.
15. $20{\cdot}67$ miles.
21. $PB = 142{\cdot}09$ yards.
42. If A is the position of the observer, the circle through the extremities of the pillar touches the horizontal at A.

XXXVI. (PAGE 210.)

1. (1) $\sqrt{\frac{5}{3}}$, $3\sqrt{\frac{3}{5}}$, $\sqrt{15}$, $3\sqrt{15}$. (2) $\frac{3}{4}\sqrt{6}$, $\frac{4}{3}\sqrt{6}$, $2\sqrt{6}$, $12\sqrt{6}$.

(3) $2\sqrt{\frac{6}{11}}$, $\sqrt{\frac{22}{3}}$, $\sqrt{\frac{33}{2}}$, $2\sqrt{66}$.

(4) $\frac{\sqrt{3}}{2}$, $\frac{7}{2\sqrt{3}}$, $7\sqrt{3}$, $\sqrt{3}$. (5) $\sqrt{2}$, $2\sqrt{2}$, $5\sqrt{\frac{1}{2}}$, $10\sqrt{2}$.

XL. (PAGE 240.)

1. $n\pi \pm \frac{\pi}{3}$; $n\pi \pm \frac{\pi}{6}$.
2. $n\pi + \frac{\pi}{4}$; $n\pi + \frac{\pi}{6}$.
3. $n\pi + \frac{\pi}{3}$; $n\pi + \frac{\pi}{4}$.
4. $n\pi \pm \frac{\pi}{4}$; $n\pi \pm \frac{\pi}{6}$.
5. $n\pi \pm \frac{\pi}{3}$; $n\pi \pm \frac{\pi}{2}$.
6. $n\pi \pm \frac{\pi}{3}$; $n\pi \pm \frac{\pi}{6}$.
7. $2n\pi \pm \frac{\pi}{3}$.
8. $2n\pi$; $2n\pi \pm \frac{\pi}{3}$.
9. $n\pi$; $2n\pi \pm \frac{\pi}{3}$.
10. $n\pi$; $n\pi \pm \frac{\pi}{6}$.
11. $n\pi + \overline{-1}|^n \frac{\pi}{3}$.
12. $n\pi + \frac{\pi}{3}$; $n\pi + \frac{\pi}{6}$.
13. $2n\pi$; $(2n+1)\frac{\pi}{2}$.
14. $n\pi$; $n\pi + \overline{-1}|^n \frac{\pi}{2}$.
15. $n\pi - \frac{\pi}{4} + \overline{-1}|^n \frac{\pi}{6}$.
16. $2n\pi$; $2n\pi + \frac{2\pi}{3}$.
17. $\sin\theta = \frac{1}{2}$ or $-\frac{1}{8}$.
18. $2n\pi$; $2n\pi - \frac{\pi}{2}$; $2n\pi + \frac{\pi}{4} \pm \frac{\pi}{2}$.
19. $n\pi + \overline{-1}|^n \frac{\pi}{6}$; $2n\pi \pm \frac{\pi}{3}$.
20. $n\pi$; $n\pi + \overline{-1}|^n \frac{\pi}{6}$.
22. $n\pi + \alpha$; $n\pi + \frac{\pi}{2} + \alpha$.
23. $\frac{n\pi}{4}$.

XLI. (Page 242.)

1. $\frac{n\pi}{3}$; $\frac{n\pi}{2} \pm \frac{\pi}{12}$.

2. $2n\pi \pm \frac{\pi}{2}$; $\frac{n\pi}{4} + \overline{-1}|^{n} \frac{\pi}{16}$.

3. $n\frac{\pi}{2}$; $n\frac{\pi}{3} + \overline{-1}|^{n} \frac{\pi}{18}$.

4. $(4n \pm 1)\frac{\pi}{3}$; $n\pi + \frac{\pi}{4}$; $2n\pi + \frac{\pi}{2}$.

5. $n\pi$; $2n\frac{\pi}{3} \pm \frac{\pi}{18}$.

6. $2n\pi - \frac{a}{2}$.

7. $\frac{n\pi}{26}$; $\frac{n\pi}{12}$.

8. $n\pi$; $\frac{2n\pi}{3} \pm \frac{\pi}{9}$.

9. $n\pi$; $\frac{n\pi}{4}$.

10. $\frac{2n\pi}{p+q}$; $\frac{(2n+1)\pi}{p}$; $\frac{(2n+1)\pi}{q}$.

11. $\frac{n\pi}{2}$; $2n\pi + \frac{\pi}{2}$; $(4n+1)\frac{\pi}{6}$.

12. $(4n \pm 1)\pi$; $n\pi$; $(2n+1)\frac{\pi}{5}$.

13. $(2n+1)\pi$; $(4n+1)\frac{\pi}{6}$.

14. $(4n \pm 1)\pi$; $\frac{2n\pi}{p} + \overline{-1}|^{n} \frac{\pi}{3p}$.

15. $2n\pi \pm \frac{\pi}{3}$; $2n\pi \pm \frac{\pi}{2}$.

16. $2n\pi + a$; $\frac{4n\pi}{3} \pm \frac{2\pi}{9} - a$.

17. $2n\pi$; $(4n+1)\frac{\pi}{6}$.

XLII. (Page 244.)

1. 0.
2. $\frac{3\sqrt{3}-7}{8}$.
3. $\frac{4+3\sqrt{15}}{20}$.
4. $\frac{2}{9}$.
5. 1.
6. $\frac{5}{9}$.
7. $\frac{156}{5}$.
8. $\sqrt{3}$; $-\sqrt{3}-2$.
9. 2.
10. $\frac{ab}{\sqrt{a^2-1}+\sqrt{b^2-1}}$.
11. $\sqrt{3}$.
12. $\frac{a-b}{1+ab}$.

XLIII. (Page 249.)

1. $\frac{n\pi}{2} \pm \frac{\pi}{8}$; $\frac{2n\pi}{3} \pm \frac{\pi}{9}$.

2. $2n\pi$; $2n\pi \pm \frac{\pi}{2}$; $\frac{2n\pi}{5}$.

3. $n\pi \pm a$.

4. $\frac{n\pi}{2} \pm \frac{\pi}{12}$.

5. $\frac{n\pi}{3}$; $\frac{n\pi}{2}$; $n\pi$.

6. $2n\pi \pm \frac{\pi}{2}$; $(2n+1)\frac{\pi}{3} + \frac{\pi}{12}$; $\frac{2n\pi}{5} - \frac{\pi}{20}$.

7. $n\pi$; $\dfrac{n\pi + \overline{-1}|^n \frac{\pi}{6}}{m-1}$. 8. $n\pi$; $n\pi - \frac{\pi}{4}$; 9. $n\pi$; $\frac{n\pi}{5}$.

10. $\frac{n\pi}{6}$; $(2n+1)\frac{\pi}{4}$. 11. $2n\pi$; $2n\pi + a$.

12. $\frac{2n\pi}{7} + \frac{\pi}{14}$; $(2n+1)\frac{\pi}{3} - \frac{\pi}{6}$. 13. $n\pi$. 14. $\frac{n\pi}{2}$.

15. $2n\pi \pm \frac{\pi}{6}$; $(2n+1)\pi - \frac{\pi}{6}$. 16. $n\pi$.

17. $2n\pi \pm \frac{\pi}{2}$; $2n\pi \pm \frac{\pi}{6}$; $2n\pi \pm \frac{\pi}{3}$. 18. $n\pi \pm \frac{\pi}{3}$; $n\pi \pm \frac{\pi}{4}$.

19. $\cos\theta = \frac{2}{3}$. 20. $2n\pi$; $n\pi + \overline{-1}|^n \frac{\pi}{3}$.

21. $(2n+1)\pi$; $2n\pi \pm \frac{\pi}{6}$; $2n\pi \pm \frac{5\pi}{6}$. 22. $\frac{n\pi}{2} + \overline{-1}|^n \frac{\pi}{4}$.

23. $n\pi \pm \frac{\pi}{2}$; $n\pi + \tan^{-1}\left(\frac{a-b}{a+b}\cot A\right)$. 24. $\frac{1}{4}(n\pi \pm \sqrt{n^2\pi^2+16})$.

25. $2x + 3y = 2n\pi \pm \frac{\pi}{3}$, $3x + 2y = 2m\pi \pm \frac{\pi}{6}$.

26. $\sin\left(\theta + \frac{\pi}{4}\right) = \frac{\pm 1}{2\sqrt{2}}$. 27. $n\pi$; $\frac{n\pi}{4}$. 28. $2\frac{n\pi}{3}$.

29. $a^2 + b^2 = 1$. 30. $\frac{n^2}{m^2} - \frac{nn'}{mm'} = 2$. 31. $(m^2 - n^2)^2 = 16mn$.

32. $x^2 + y^2 = 2a^2$. 33. $\frac{a^2}{b^2} + \frac{a}{2c} = 1$.

35. $2x = \sqrt{(b+c)^2 \sec^2 a + (b-c)^2 \operatorname{cosec}^2 a}$.

37. $\left(\frac{x^2+y^2-1}{3}\right)^{\frac{1}{2}} + 1 = 2\left(\frac{y}{2}\right)^{\frac{2}{3}}$.

39. $\{m+n+1-(m-n)^2\}^2 + (m-n)^2 = 2$.

MISCELLANEOUS EXAMPLES.

1. 30°, 25°, 125°. 2. 89° 58′ 48″. 3. 6°.

4. 105°, 60°, 15°. 6. That number of sides is the same in both. 8. 50° 6′ nearly.

10. $\tan A = \frac{p}{q}\sqrt{\frac{q^2-1}{1-p^2}}$, $\tan B = \sqrt{\frac{q^2-1}{1-p^2}}$.

12. (1) $n\pi+(-1)^n\frac{\pi}{2}$, $\sin^{-1}\left(-\frac{3}{5}\right)$.

(2) $n\pi+(-1)^n\frac{\pi}{6}$, (3) $n\pi+\frac{\pi}{3}$, $n\pi+\frac{\pi}{6}$.

13. $\theta=n\pi\pm\alpha$, $\theta=2n\pi+\frac{3\pi}{4}$. **46.** $x=2$.

61. (1) $\theta=2n\pi$ or $2n\pi+\frac{\pi}{2}$, (2) $n\pi\pm\frac{\pi}{6}$, (3) $\frac{n\pi}{2}\pm\frac{\pi}{12}$,

(4) $\theta=\frac{m+n}{2}\pi+(-1)^m\frac{\pi}{4}+(-1)^n\frac{\pi}{12}$,

(5) $\theta=n\pi\pm\frac{\pi}{2}$, $n\pi\pm\frac{\pi}{4}\pm\frac{\pi}{20}$, $n\pi\pm\frac{\pi}{10}$, (6) $\theta=(2n+1)\frac{\pi}{4}$.

66. $a^2+c^2-2ac\cos 2\phi=b^2$. **70.** $x=0$, $y=\frac{\sqrt{3}}{2}$.

72. $\frac{1}{2}\left\{n-\frac{\cos(n+1)\alpha\sin n\alpha}{\sin\alpha}\right\}$.

73. $\frac{1}{4}\sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2}\left(\sin\frac{n+3}{2}x+\sin\frac{n-1}{2}x+\sin\frac{n+7}{2}x\right)$

$-\frac{1}{4}\sin\frac{3nx}{2}\operatorname{cosec}\frac{3x}{2}\sin\frac{3n+9}{2}x$.

74. $\frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}}-2\cot 2x$.

83. $A=85°\,54'\,15''$, $B=18°\,9'\,31''$ nearly.

84. $B=28°\,14'\,50''$ nearly.

85. $78°\,27'\,46''$, $A=90°$, $B=60°$. **91.** 66 feet.

92. 314·159 sq. inches. **103.** 8 feet nearly.

104. Ratio of areas is $2+\sqrt{3}:2$. **114.** 25·7834.

115. Sides are 7, 8, 5, angles $38°\,12'\,48''$, $98°\,12'\,48''$.

120. ·535.

EXAMINATION PAPERS.

I. (1) 15. (2) 9, $16\frac{2}{3}$. (3) 44 feet. (6) 34 feet nearly. (7) $114°\,35'$ nearly.

II. (1) 1200°. (3) For $A=0°$, 30°, 45°, 60°, 90° the values are 1, $\frac{1}{2}(\sqrt{3}+1)$, $\sqrt{2}$, $\frac{1}{2}(\sqrt{3}+1)$, 1. (5) $\frac{2\sqrt{2}}{3}$, $2\sqrt{2}$. (6) $\frac{1}{6}(2\sqrt{6}+1)$. (7) $2\sin 35°\cos 30°$.

III. (1) 314·59 sq. feet. (2) $\frac{\pi}{2}, \frac{3\pi}{5}, \frac{2\pi}{3}$. (3) 9.

(5) For the values 0°, 30°, 45°, 60°, 90° we have $-\infty, -\frac{2}{\sqrt{3}}, 0, \frac{2}{\sqrt{3}}, \infty$.

(6) $60°, \frac{\pi}{3}, 60^g$; $60°, \frac{\pi}{3}, 50^g$. (8) $\frac{5\sqrt{15}-\sqrt{11}}{24}$.

IV. (1) $\frac{\pi}{3}$, 1° 40′. (4) $a=4\sqrt{3}$, $\cos B=\frac{4\sqrt{3}}{13}$.

(5) $100\sqrt{3}$ feet. (7) $\frac{1}{2}\{\cos(A-B)-\cos(A+B)\}$,

$\frac{1}{2}\{\cos 75°+\cos 45°\}$, $\frac{1}{2}\{\sin 2\alpha+\sin 2\beta\}$. (8) $\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$.

V. (1) 70°. (4) 1, +1, cos 50°, − cos 70°, cot 30°. (5) $\frac{2}{11}$.

VI. (3) $\sin A$, $-\cot A$, 0.

VII. (1) 15°. (2) 40^g. (4) cosine is $\frac{4}{5}$, tangent is $\frac{3}{4}$.

(7) 1. (8) 1129·1.

VIII. (1) 13·7 nearly. (2) $\frac{84}{85}$. (5) 16296 sq. ft.

(6) $2n\pi \pm \frac{2\pi}{3}$. (7) $\frac{12}{13}$. (8) $B=29° 17' 32''$ nearly.

IX. (4) $\frac{2-\sqrt{6}}{2\sqrt{5}}$.

X. (2) $50\sqrt{3}$ yards. (3) $\frac{36}{85}$. (7) $\theta=n\pi\pm\frac{\pi}{4}$.

XI. (2) $-\frac{99}{20}$. (7) $\frac{9}{2}$.

XII. (3) $\frac{5\pi}{4}, -\frac{\pi}{4}; \frac{3\pi}{4}, -\frac{\pi}{4}$.

(5) $\frac{n\pi}{2}+\frac{\pi}{8}, \frac{n\pi}{4}+(-1)^n\frac{\pi}{12}$.

XIV. (7) 109° 6′ 24″, 10° 53′ 36″.

XV. (1) 45°, 75°, 105°, 135°; $\frac{\pi}{4}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}$.

(2) $\pm\frac{28}{53}, \pm\frac{45}{28}$. (4) $\sin A=\frac{1}{3}$ is one solution.

XVI. (1) 3·14159 feet. (2) $n\pi \pm a$.

XVII. (2) $\theta = n\pi \pm \frac{\pi}{10}$. (5) 64° 27′, 52° 57′. (6) 120 feet.

XVIII. (1) $\frac{55}{108}$. (3) $\frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$.

(5) $A = 56° 34' 15''$, $C = 85° 25' 45''$ or $A = 123° 25' 45''$, $C = 18° 34' 15''$.

(7) $(2n+1)\pi$, $n\pi \pm \frac{\pi}{6}$.

XIX. (3) $\frac{2n\pi}{3}, \frac{(2n+1)\pi}{5}$.

XX. (2) $\pm\frac{3}{5}, \frac{24}{7}$. (4) 89° 10′ 10″. (8) (1) $2n\pi$, $2n\pi - \frac{2\pi}{3}$,

(2) $2n\pi + \frac{\pi}{2} - a$, $(2n-1)\pi + a$.

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